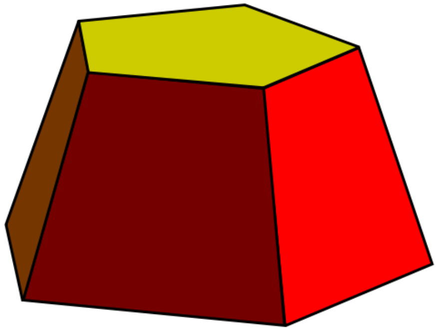
Triple Integration

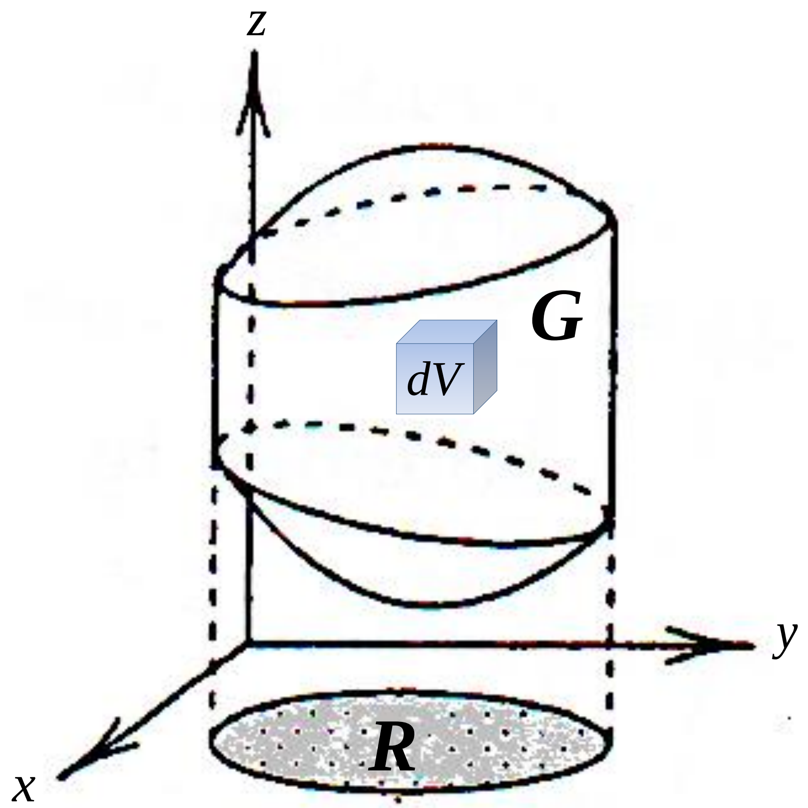
- Volume of the objects
- Mass, moment of inertia
- Impact of any scalar field on the objects

$$\iiint$$

Motivation:



Idea:



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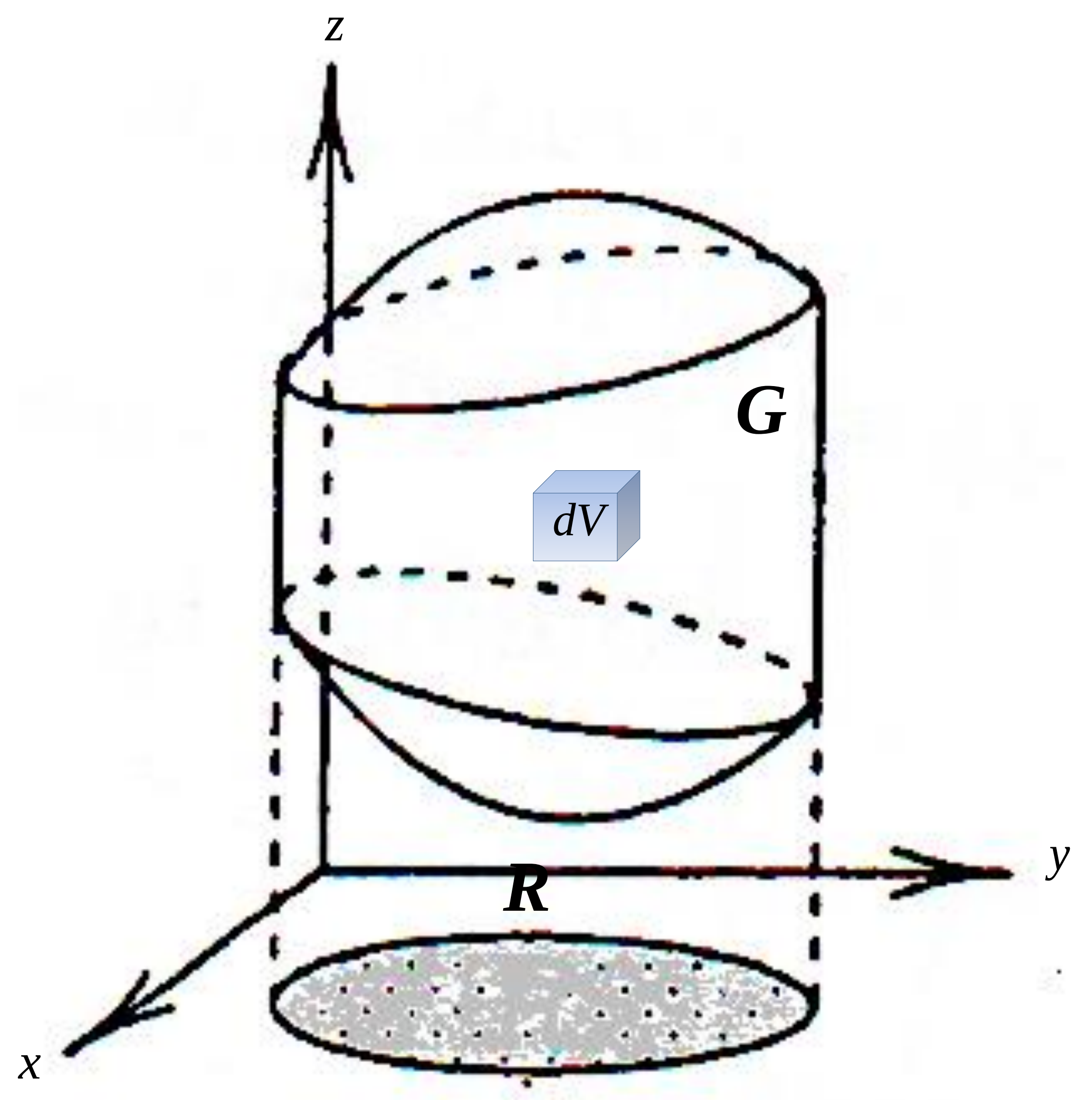
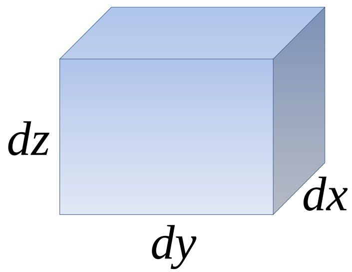


Idea:

Volume of a small element is



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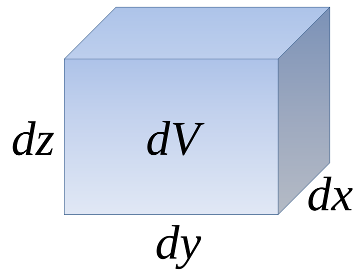

$$dV = dz\,dx\,dy = dz\,dA$$

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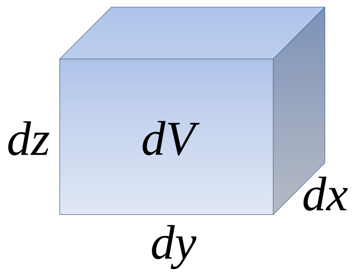


$$dV = dzdxdy = dzdA$$

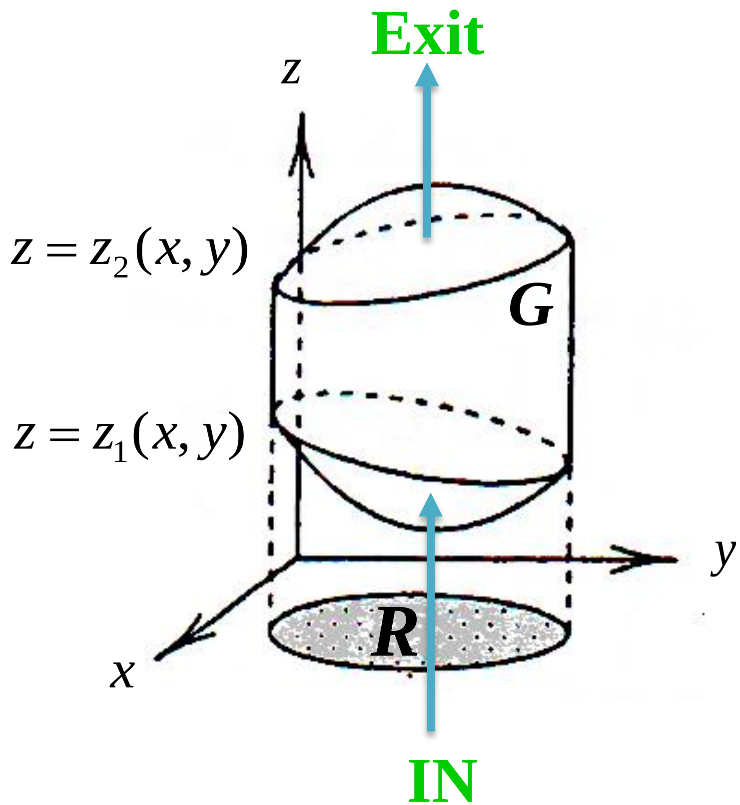
How to get total volume enclosed by G?

Integrate over the entire surface !!!

Idea:

Volume of a small element

$$dV = dzdydx = dzdA$$



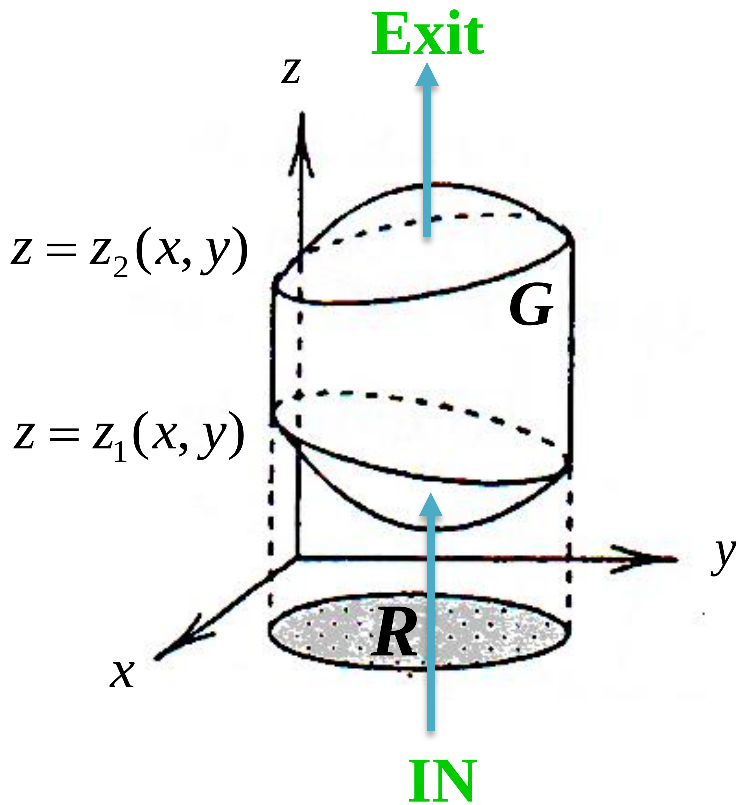
Idea:

Volume of a small element

$$dV = dzdydx = dzdA$$

Total volume of a 3-D closed region G

$$V = \iiint_G dV$$



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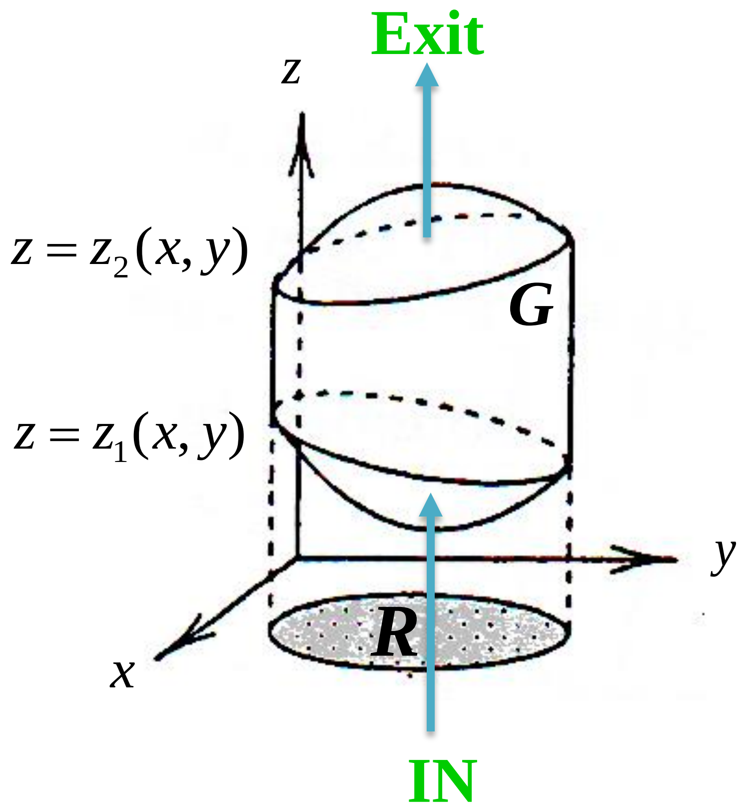

Triple integrals – Cartesian coordinates

Volume of a small element

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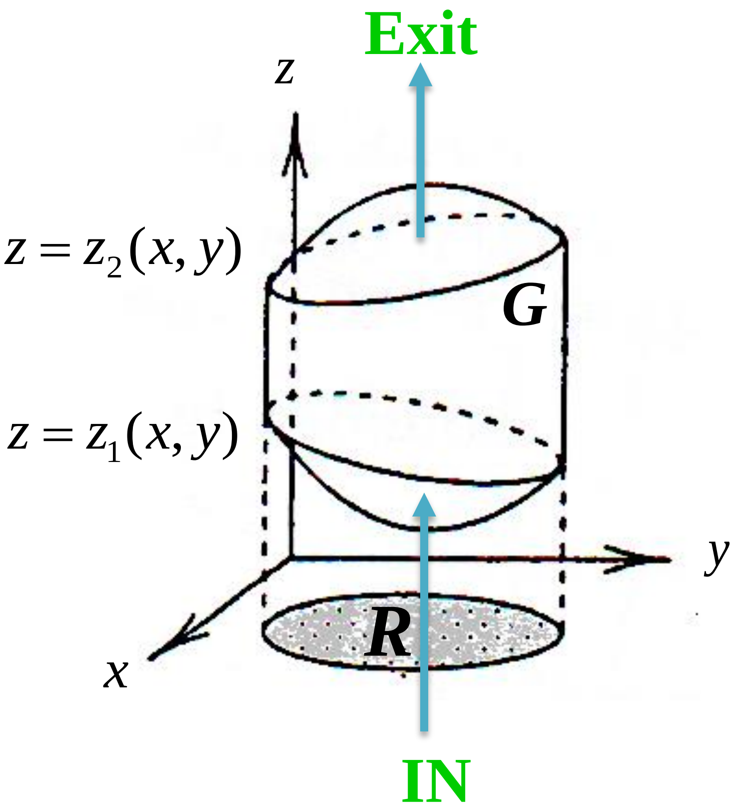

Applications

A function of three variables $f(x,y,z)$ can be interpreted as a field that varies at each point (x,y,z).

Examples:

Pressure $P = P(x,y,z)$
Temperature $T = T(x,y,z)$

Applications

A function of three variables *f* (*x*,*y*,*z*) can be interpreted as a field that varies at each point (*x*,*y*,*z*).

Examples:

Pressure $P = P(x,y,z)$

Temperature $T = T(x,y,z)$

How to calculate impact of *f* (*x*,*y*,*z*) on G?

$$\text{Impact} = \iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dA$$

Practice-1: Find the volume of the solid bounded by

$x^2 + y^2 = 9$, $y + z = 4$ and $z = 0.$

The region G and its projection R on the xy - plane is as shown below

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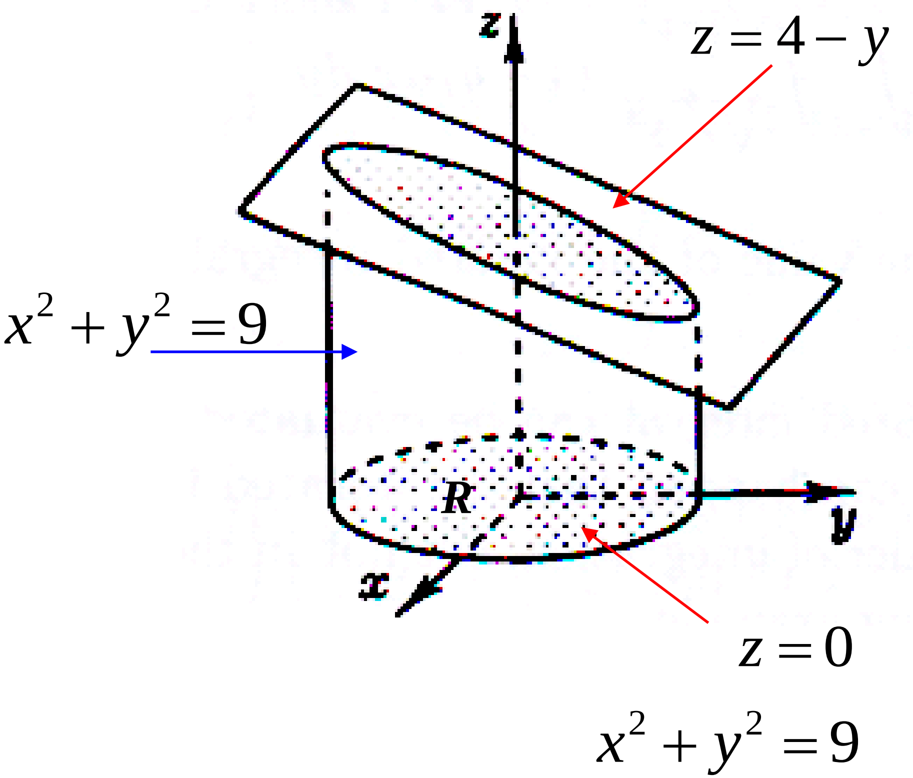
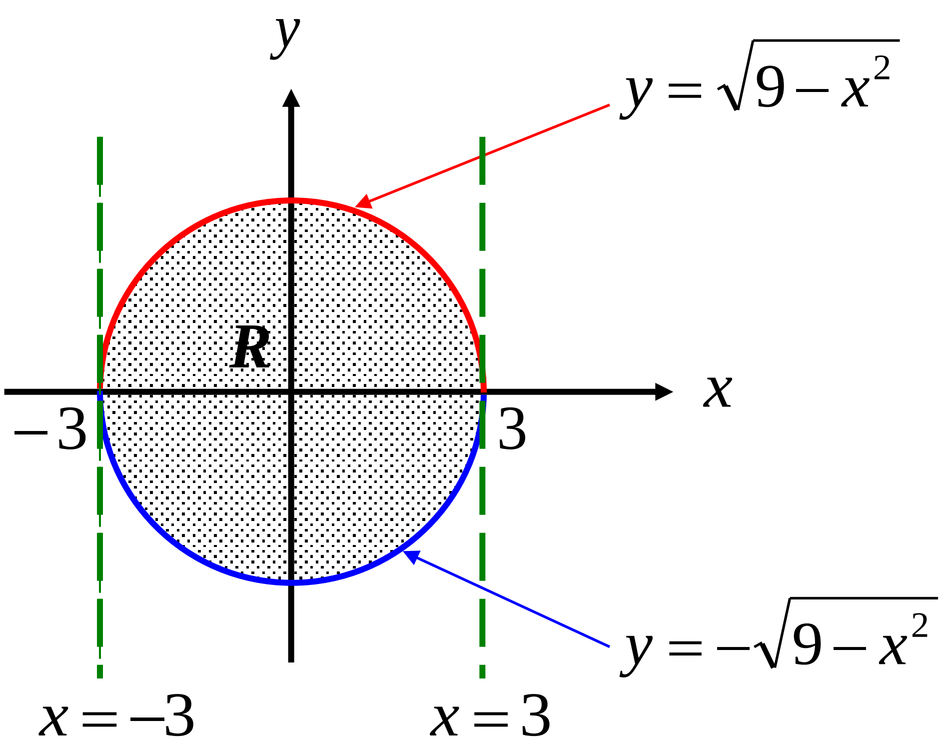
Solution





Practice-1:

Solution

$$V=\iiint_G dV=\iint_R\left[\int_{z=0}^{z=4-y}dz\right]dA$$

$$=\iint_R(4-y)dA=\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}(4-y)\,dy\,dx$$

$$=\int_{-3}^{3}\left[4y-\frac{y^2}{2}\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}dx$$

$$=8\int_{-3}^{3}\sqrt{9-x^2}\,dx$$

Practice-1:

Now let $x = 3\sin\theta, dx = 3\cos\theta d\theta$.

Solution

Hence

$$V = 8\int_{-3}^{3}\sqrt{9-x^2}\,dx = 8\int_{-\pi/2}^{\pi/2}\sqrt{9(1-\sin^2\theta)}\,(3\cos\theta d\theta)$$

$$= 72\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = 36\int_{-\pi/2}^{\pi/2}(\cos 2\theta + 1)d\theta$$

$$= 36\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\pi/2}^{\pi/2}$$

$$= 36\left[\left(\frac{\sin\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin(-\pi)}{2} - \frac{\pi}{2}\right)\right] = 36\pi$$

Practice-1:

Another way …

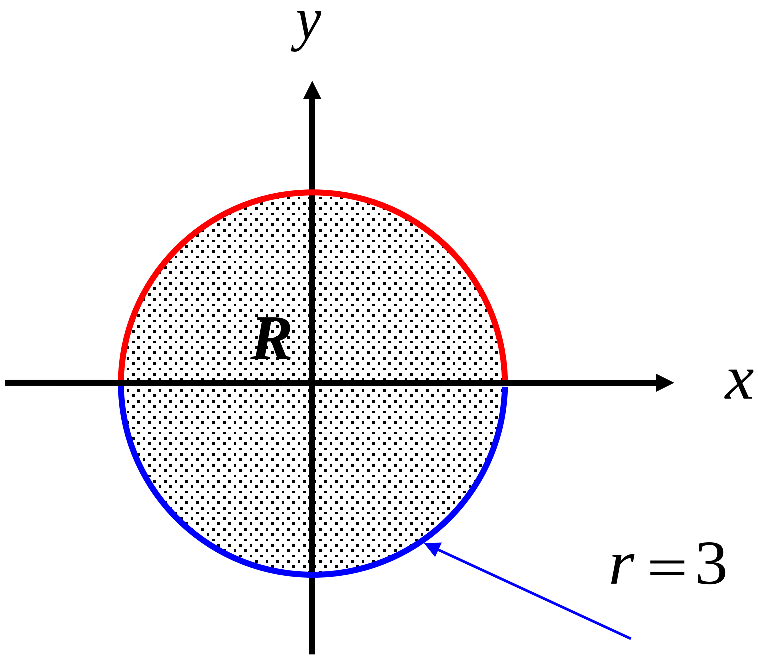
$$V=\iiint_G dV=\iint_R\left[\int_{z=0}^{z=4-y}dz\right]dA$$

$$=\iint_R (4-y)dA$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3}(4-r\sin\theta)rdrd\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{r^3}{3}\sin\theta\right]_0^3 d\theta=9\int_0^{2\pi}(2-\sin\theta)d\theta$$

$$=9(2\theta+\cos\theta)_0^{2\pi}=36\pi$$



Practice-2: If G is the region in the first octant bounded by $y = x^2, z + y = 1, xy$-plane and yz-plane, evaluate

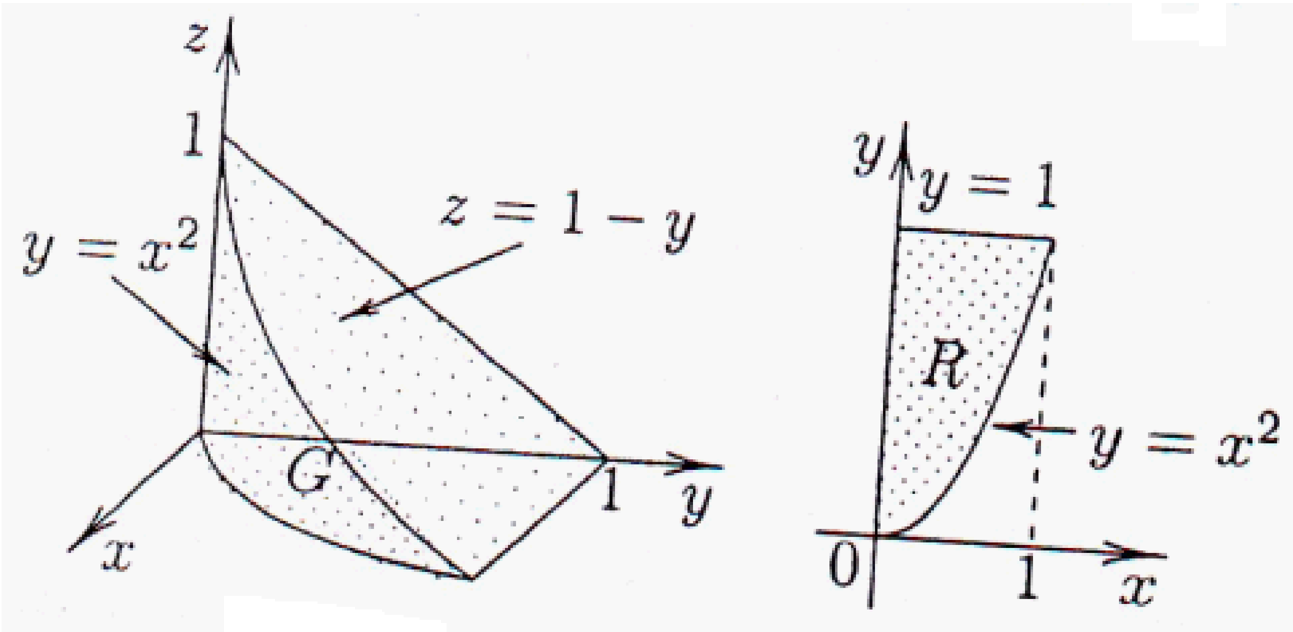
(i) $\iiint_G 6z\, dV$ (ii) The volume of the region G

Practice-2: If G is the region in the first octant bounded by $y = x^2, z + y = 1, xy$-plane and yz-plane, evaluate

(i) $\iiint_G 6z \, dV$ (ii) The volume of the region G

Solution

The region G and its projection R on the xy-plane is as shown below



(i) In this case we have $z_1(x,y) = 0$ and $z_2(x,y) = 1 - y$. The projection of G on the xy-plane is a region R bounded by $x = 0, x = 1, y = x^2$ and $y = 1$.

Practice-2:

Thus we obtain

Solution

$$\iiint_G f(x,y,z)\,dV = \iiint_G 6z\,dV = \iint_R \left[\int_{z=0}^{z=1-y} 6z\,dz\right] dA$$

$$= \iint_R \left[3z^2\right]_{z=0}^{z=1-y} dA$$

$$= \iint_R 3(1-y)^2\,dA$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} 3(1-y)^2\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left[-(1-y)^3\right]_{y=x^2}^{y=1} dx$$

$$= \int_0^1 (1-x^2)^3\,dx = \int_0^1 (1-3x^2+3x^4-x^6)\,dx$$

$$= \left[x - x^3 + \frac{3}{5}x^5 - \frac{x^7}{7}\right]_0^1 = \frac{16}{35}$$

Practice-2:

(ii)

Solution

$$V=\iiint_G dV=\iint_R\left[\int_{z=0}^{z=1-y}dz\right]dA=\iint_R(1-y)\,dA$$

$$=\int_{x=0}^{x=1}\int_{y=x^2}^{y=1}(1-y)\,dy\,dx$$

$$=\int_{x=0}^{x=1}\left[y-\frac{y^2}{2}\right]_{y=x^2}^{y=1}dx$$

$$=\int_0^1\left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx=\left[\frac{x}{2}-\frac{x^3}{3}+\frac{x^5}{10}\right]_0^1=\frac{4}{15}$$

Practice-3: Find the volume of the solid bounded by

$x^2 + y^2 = 9, z = \sqrt{25 - x^2 - y^2}$, and $z = 0$.

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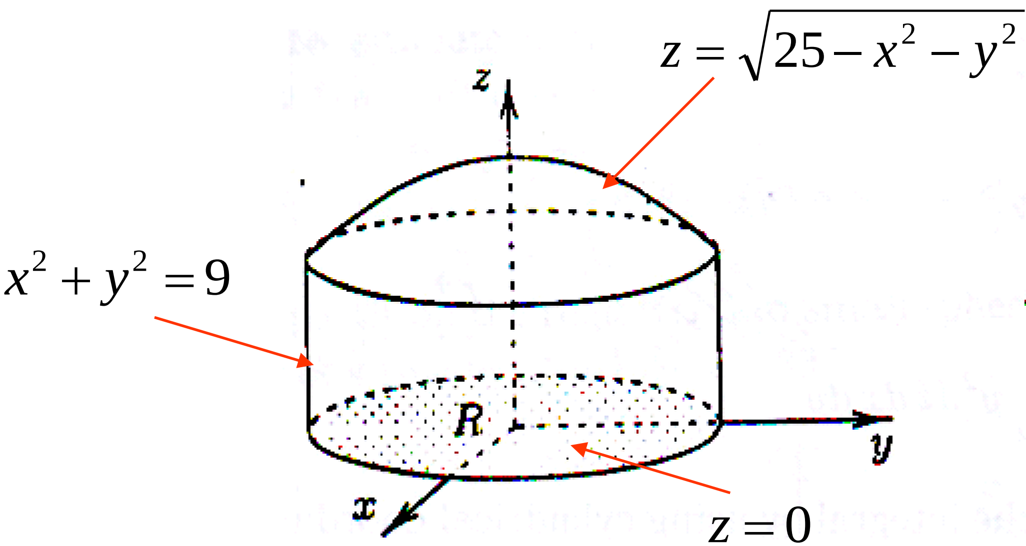

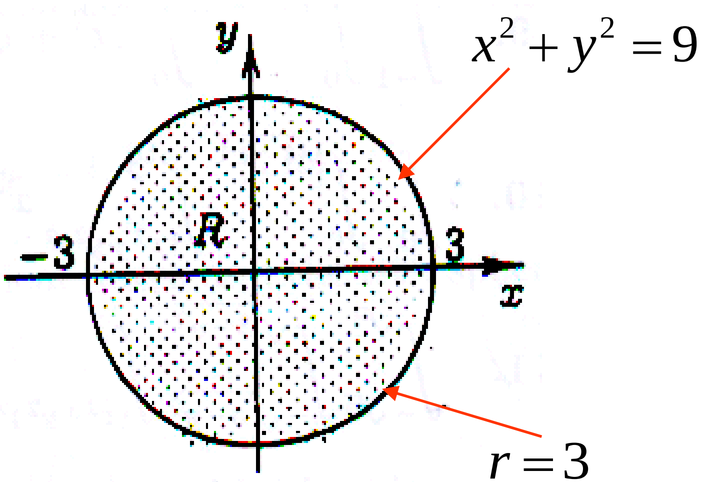

Practice-3:

Solution

$$V = \iiint_G dV = \iint_R \left[\int_{z=0}^{z=\sqrt{25-x^2-y^2}} dz \right] dA$$

$$= \iint_R \sqrt{25 - x^2 - y^2}\, dA$$

Now it will be easy to convert double integral into polar coordinates

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} \left(\sqrt{25 - r^2} \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{-(25 - r^2)^{3/2}}{3} \right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{61}{3}\, d\theta = \frac{122\pi}{3}$$

Practice-4: Use cylindrical coordinates to find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

Solution

The solid of the region G and its projection R on the xy - plane are

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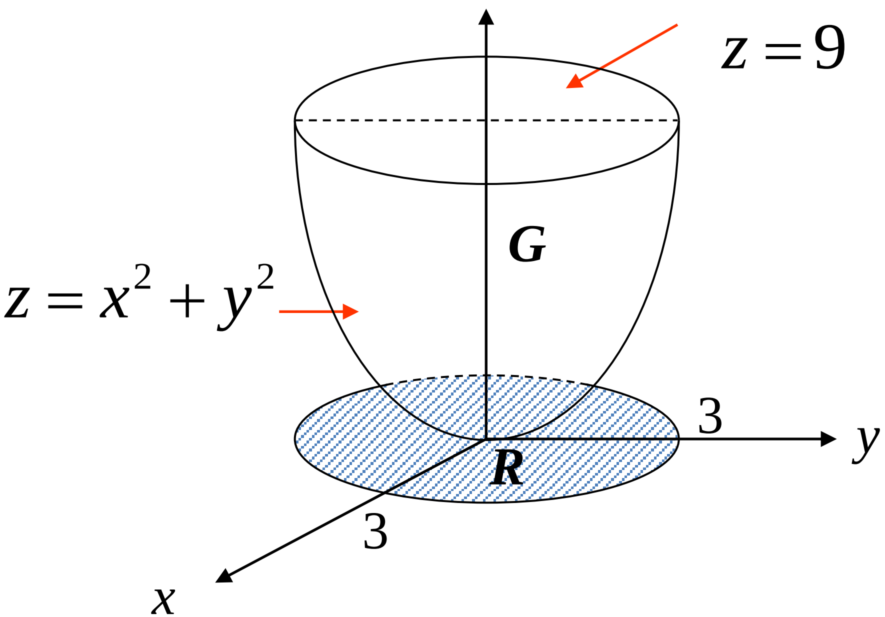

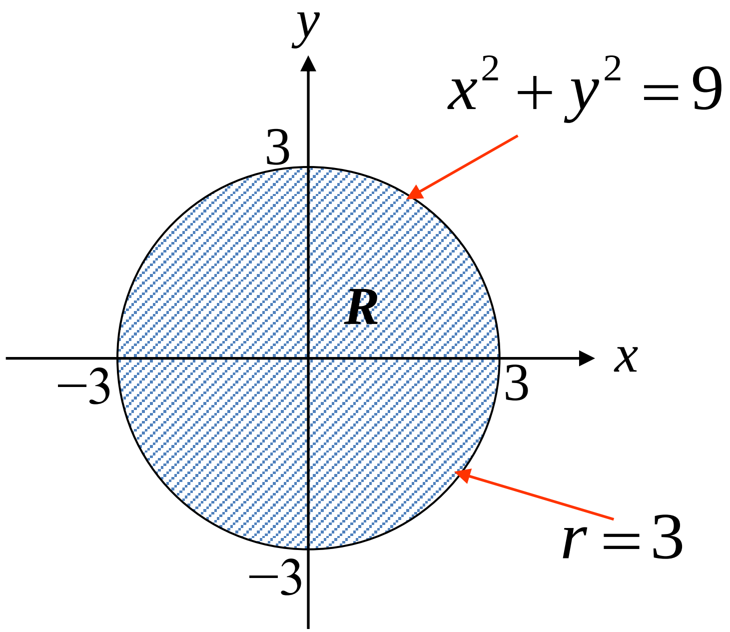

Practice-4:

Solution

Thus the required volume is

$$V = \iiint_G dV = \iint_R \left[\int_{z=x^2+y^2}^{z=9} dz \right] dA$$

$$= \iint_R (9 - x^2 - y^2)\, dA$$

Now lets convert above double integral into polar coordinate s

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\, d\theta = \frac{81\pi}{2}$$