Fourier Series

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contains terms on the right which are all odd polynomials. Similarly cos −series comprise of all even polynomials.

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For example, the series of

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contains terms on the right which are all odd polynomials. Similarly $\cos -$series comprise of all even polynomials.

Can we generalize this idea to incorporate functions that share other properties?

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Such functions must be expressed in terms of fundamental functions that share repetitive behavior. We know that both sin & cos are functions that repeat themselves (periodic), therefore, it would be interesting to study a series that contains both of them.

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What about functions that exhibit repetitive behavior? Such functions must be expressed in terms of fundamental functions that share repetitive behavior. We know that both sin & cos are functions that repeat themselves (periodic), therefore, it would be interesting to study a series that contains both of them.

Such a series is known as Fourier series, named after a French mathematician Joseph Fourier (1768-1830), who introduced it during his study of heat flow.

Fourier Series

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Why is it important to divide any continuous or discontinuous function in terms of oscillations?

To answer this ask yourself another question. How does your cell phone receive or send signals in any kind of transmission? The answer is that a communication takes place when both receiver and transmitter interact each other via basic oscillatory modes of particles.

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To answer this ask yourself another question. How does your cell phone receive or send signals in any kind of transmission? The answer is that a communication takes place when both receiver and transmitter interact each other via basic oscillatory modes of particles.

Your course "Signals and Systems" is also based on this simple idea !!!.

Fundamental Modes

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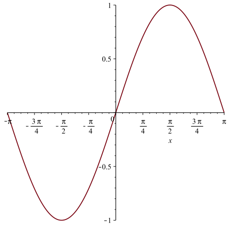
What are the periods of $\sin(2x)$,$\tan(x)$ and $\cos(5x)$?

Fundamental Modes

Graph of "sin(x), sin(2x), sin(3x)" function for different frequency modes in the interval $[-\pi, \pi]$.



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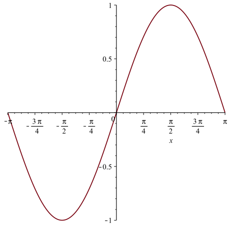

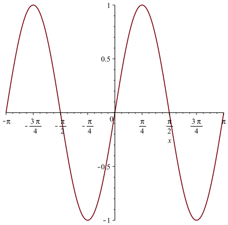

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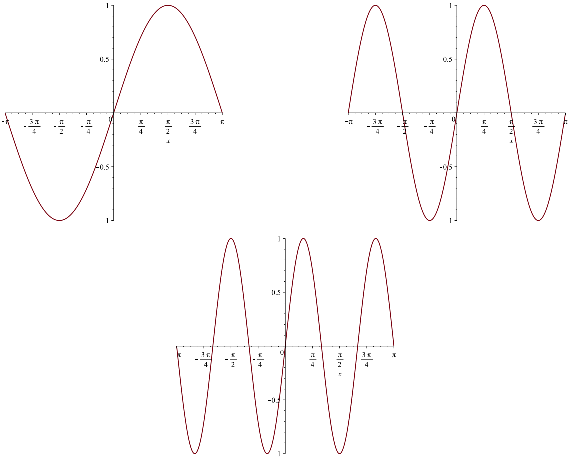
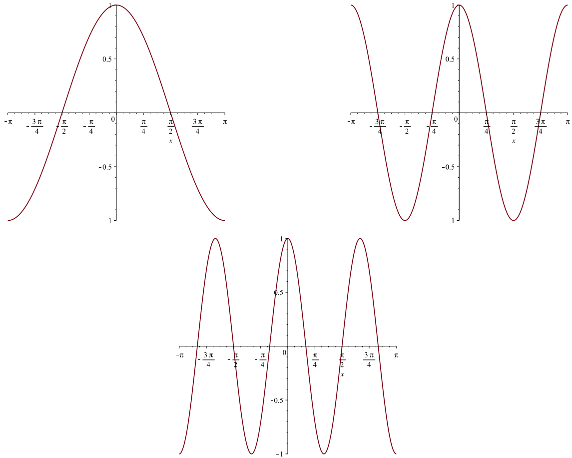


Fundamental Modes

Graph of “cos(x), cos(2x), cos(3x)” function for different frequency modes in the interval $[-\pi, \pi]$.



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$$f(x) = a_0 + \underbrace{a_1 \sin kx + b_1 \cos kx}_{\text{first mode}} + \underbrace{a_2 \sin 2kx + b_2 \cos 2kx}_{\text{second mode}} + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \sin(nkx) + b_n \cos(nkx)\right)$$

where $k = \pi/L$ and the coefficients are determined in three steps.

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where $k = \pi/L$ and the coefficients are determined in three steps.

Step-1 $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Step-2 $$a_n = \frac{1}{L}\int_{-L}^{L} \sin(nkx) f(x)\,dx$$

Step-3 $$b_n = \frac{1}{L}\int_{-L}^{L} \cos(nkx) f(x)\,dx$$

Fourier Series

Clarification about k.

Since the second term in Fourier series is $\sin kx$ therefore its period is $2\pi/k$ but the period of given function was $2L$ therefore we must have

$$\frac{2\pi}{k} = 2L \qquad \Rightarrow k = \frac{\pi}{L}$$

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Step-1 $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Step-2 $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x)\, dx$$

Step-3 $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x)\, dx$$

Fourier Series

Q: To which functions we can apply Fourier series?

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Ans: It can be applied to both continuous and discontinuous functions. However the convergence of a series at a particular point may or may not be equal to the value of the function at that point. For most functions arise in engineering problems this is not the case.

Formulae

There are few formulae that you would need to remember.

1. $\sin(n\pi) = 0 = \cos\left((2n+1)\frac{\pi}{2}\right)$
2. $\sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n = \cos(n\pi) \quad n = 0, 1, 2, \ldots$
3. For odd fn. $f(-x) = -f(x)$, $\displaystyle\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\, dx = 0$
4. For even fn. $f(-x) = f(x)$, $\displaystyle\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\, dx = 2\int_{0}^{L} f(x)\, dx$

In the light of above formulae we can deduce that for an odd function we have $a_0, b_n = 0$ and for even functions $a_n = 0$.

Examples

Q.1: Find the Fourier series of $\sin(3x)$, $\cos 5x$, $2\sin(2x)$?

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Q.2: Find the Fourier series of x for a period $2L = 2\pi$?

Q.3: Find the Fourier series of $f(x)$ for a period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Solution

Example-1: Find the Fourier series of x for a period $2L = 2\pi$?

Ans. Since $f(x)$ is an odd function therefore it is natural to see $b_n = 0$ for all n, because $\cos(nx)$ is an even function thus we only need to calculate

Step-2

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx \quad \text{Integration by parts!!!} \\
&= \frac{1}{\pi}\left(\frac{-x\cos nx}{n}\right)_{x=-\pi}^{x=\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\,dx \\
&= \frac{2}{n}(-1)^{n+1} - 0 \quad \text{Reason !!!}
\end{aligned}
$$

Verify that $a_0 = 0$.

Solution

Example-1: Find the Fourier series of x for a period $2L = 2\pi$?

Ans. Therefore

$$f(x) = 2\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{\sin nx}{n}$$
$$= 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

Use the value $x = \pi/2$ to see an interesting fact !!!

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right)$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

which is a series representation of $\pi/4$.

Geometrical Understanding

1. The function $f(x) = x, \quad -\pi < x < \pi$ has the Fourier series representation

$$f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right)$$

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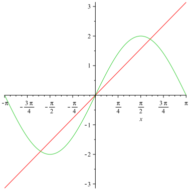

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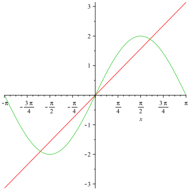

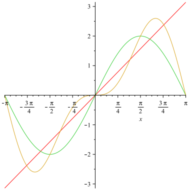

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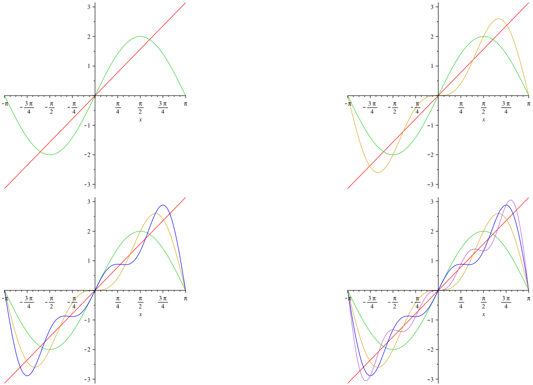
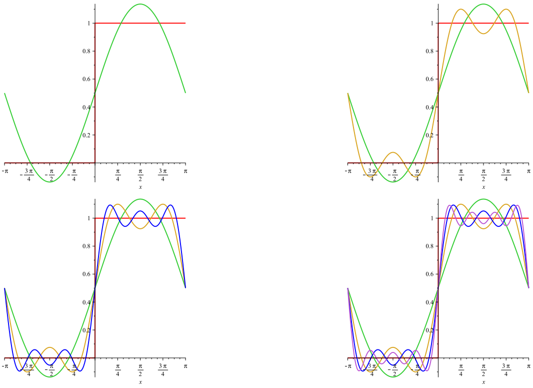


Geometrical Understanding

2. The function $f(x) = 1, \quad -\pi < x < 0$ and $f(x) = 0, \quad 0 < x < \pi$ has the Fourier series representation

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$



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Ans.

Step-1 $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} 1\,dx + 0 \quad \text{(Reason !!!)} \qquad = \frac{1}{2}$$

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Step-2 $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos nx}{n}\right)_{x=-\pi}^{x=0} = \frac{1}{n\pi}\left(-1+(-1)^n\right)$$

Solution

Example-2: Find the Fourier series of $f(x)$ for a period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Ans. Step-2

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left(-1 + (-1)^n\right) \\ &= \frac{-2}{n\pi} \quad \text{if } n \text{ is odd and zero otherwise} \end{aligned}$$

Solution

Example-2: Find the Fourier series of $f(x)$ for a period 2π such that

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Ans. Step-3

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx \\ &= \frac{1}{\pi}\left(\frac{-\sin nx}{n}\right)_{x=-\pi}^{x=0} \\ &= 0 \end{aligned}$$

Solution

Example-2: Find the Fourier series of $f(x)$ for a period 2π such that

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Ans. Step-3

Therefore,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{n=\infty} \left(\frac{-2}{\pi}\right) \frac{\sin nx}{n}$$
$$= \frac{1}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

Practice:

Q.4: Find the Fourier series of x^2 for a period $L = 2\pi$?

Ans. Since $f(x)$ is an even function therefore it is natural to see $a_n = 0$ for all n, because $\sin(nx)$ is an odd function. Follow the steps as in the earlier questions.

Also find the series of $\pi^2/6$.

Practice:

Q.5: Find the Fourier series of $f(x)$ for a period 2π such that

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Practice:

Q.6: Find the Fourier series of $f(x)$ such that

$$f(x) = 1 - x^2, \quad (-1 < x < 1)$$

Practice:

Q.7: Find the Fourier series of $f(x)$ such that

$$f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

Practice:

Q.8: A sinusoidal voltage $E\sin\omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0, & -L < t < 0 \\ E\sin\omega t, & 0 < t < L \end{cases}$$

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Ans. Note that here $2L = 2\pi/\omega \Rightarrow L = \pi/\omega$. The answer is

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right)$$

Complex Fourier Series:

Using the well-known Euler formula $e^{ix} = \cos x + i \sin x$, a complex Fourier series can be obtained

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$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/L},$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

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The reasons for using complex Fourier series are following:
1. It helps to combine both "sin" and "cos" in a unified unit "e^{ix}".
2. Both coefficients a_n and b_n can be obtained easily from complex coefficients c_n.

Example

Q.1: Find the complex Fourier series of $f(x) = x, \quad -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

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Ans. Note that here $2L = 2\pi \Rightarrow L = \pi$, so for $n \neq 0$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx,$$

$$2\pi c_n = \left(\frac{xe^{-inx}}{-in}\right)_{x=-\pi}^{x=\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$2\pi c_n = \frac{\pi}{-in}\left(e^{-in\pi} + e^{in\pi}\right) + 0 \text{ Reason !!!}$$

$$2\pi c_n = \frac{\pi}{-in}\left(2\cos n\pi\right)$$

$$2\pi c_n = \frac{2i\pi(-1)^n}{n}$$

$$\Rightarrow c_n = \frac{i(-1)^n}{n}$$

Example

Q.1: Find the complex Fourier series of $f(x) = x, \quad -\pi < x < \pi$ and $f(x+2\pi) = f(x)$ and obtain from it the usual Fourier series.

Ans. For $n = 0$,

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x)\,dx \\ &= 0 \end{aligned}$$

Also it is an odd function.

Example

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Ans.

$$f(x) = \sum_{n=-\infty}^{n=\infty} \frac{i(-1)^n}{n} e^{inx}$$
$$= \ldots - \frac{1}{2} ie^{-2ix} + \frac{1}{1} ie^{-ix} - \frac{1}{1} ie^{ix} + \frac{1}{2} ie^{2ix} + \ldots$$

Example

Q.1: Find the complex Fourier series of $f(x) = x, \quad -\pi < x < \pi$ and $f(x+2\pi) = f(x)$ and obtain from it the usual Fourier series.

Ans.

$$\begin{aligned} f(x) &= \dots - \frac{1}{2} ie^{-2ix} + \frac{1}{1} ie^{-ix} - \frac{1}{1} ie^{ix} + \frac{1}{2} ie^{2ix} + \dots \\ &= \dots - \frac{1}{2} i(\cos 2x - i \sin 2x) + \frac{1}{1} i(\cos x - i \sin x) \\ &- \frac{1}{1} i(\cos x + i \sin x) + \frac{1}{2} i(\cos 2x + i \sin 2x) - \dots \\ &= 2 \sin x - \sin 2x + \dots \\ &= 2\left(\sin x - \frac{\sin 2x}{2} + \dots\right) \end{aligned}$$

Practice

Q.2: Find the complex Fourier series of $f(x) = e^x, \quad -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Practice

Q.3: Find the complex Fourier series of $f(x) = x^2, \quad -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Convergence

Although Fourier series can be obtained for a large class of functions. However, it can not be found for functions which carry abrupt changes in short time. Therefore, there are three types of conditions known as Dirichlet conditions to be satisfied.

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Condition-1 Over any period, $f(x)$ must be absolutely integrable, i.e.,

$$\int_X |f(x)|\, dx < \infty$$

Counter example: $f(x) = 1/x, \quad 0 < x < 1.$

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Condition-2 In any finite interval, $f(x)$ is of bounded variation, i.e., there are no more than a finite number of maxima and minima in that interval.

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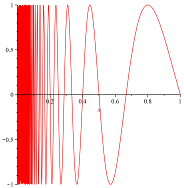

Convergence

Although Fourier series can be obtained for a large class of functions. However, it can not be found for functions which carry abrupt changes in short time. Therefore, there are three types of conditions known as Dirichlet conditions to be satisfied.

Condition-2 In any finite interval, $f(x)$ is of bounded variation, i.e., there are no more than a finite number of maxima and minima in that interval.

Counter example: $f(x) = \sin(2\pi/x), \quad 0 < x \leq 1.$

Condition-3 In any finite interval, $f(x)$ is of bounded variation, i.e., there are finite number of discontinuities.

Counter example: Draw a function.