Extreme Values

&

Saddle Points

Absolute Maximum and Minimum

- A function $f(x,y)$ has an **absolute maximum** (or global maximum) at (a,b) if

$$f(a,b) \geq f(x,y) \quad \forall \ (x,y) \in D$$

The number $f(a,b)$ is called maximum value of f in D.

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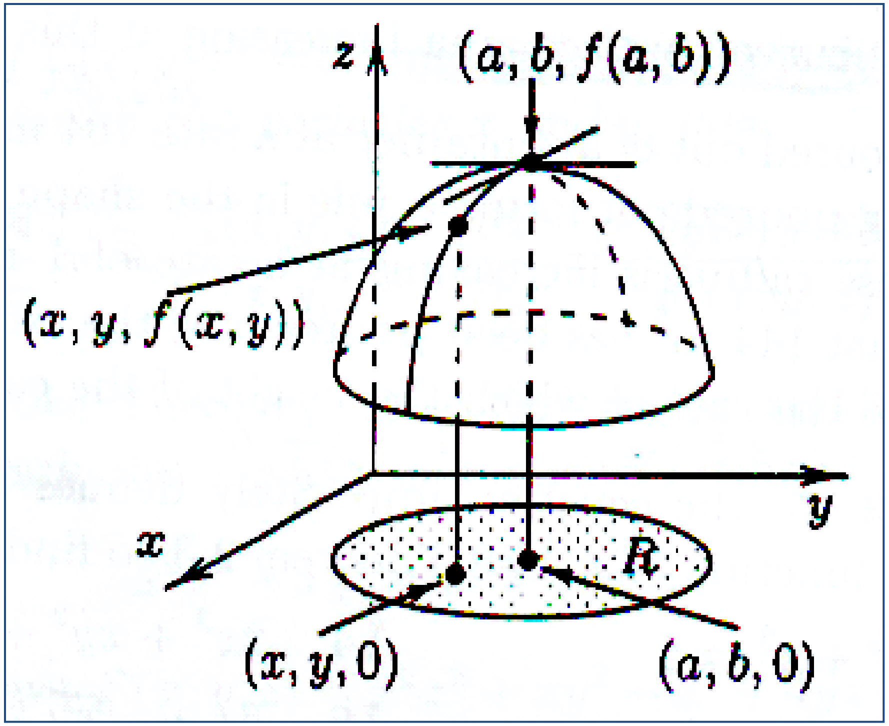
Local Maximum

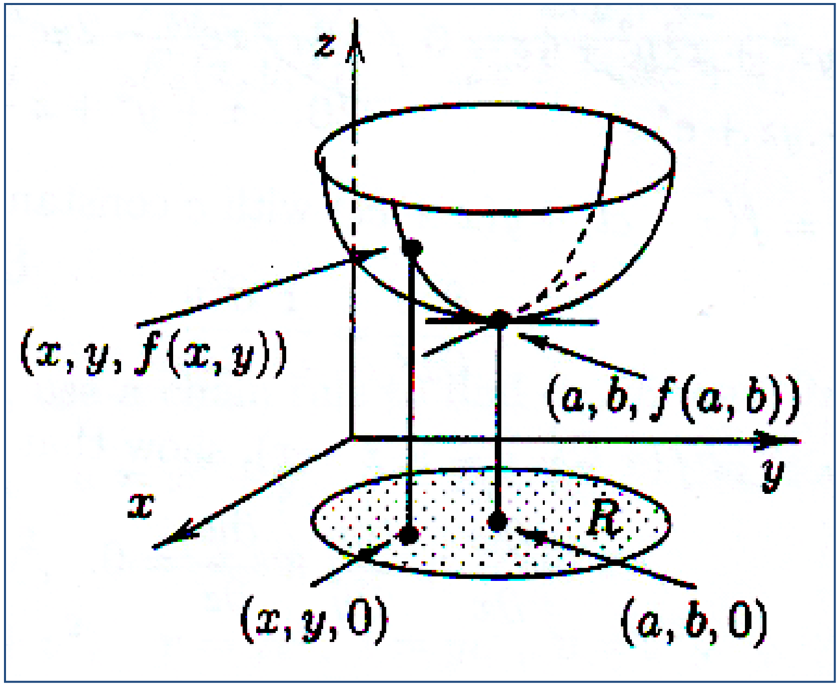
Geometrically:



Point $(a, b, f(a,b))$ is a (local) maximum

Local Minimum

Geometrically:



Point $(a,b, f(a,b))$ is a (local) minimum

Critical Points

An interior point of the domain of a function $f(x,y)$ where:

- Both f_x and f_y are **zero**

Remark:

Local Extrema exist only at the critical points or at the boundary points of the domain

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An interior point of the domain of a function $f(x,y)$ where:

- Both f_x and f_y are **zero**

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- Where one or both of f_x and f_y **fail** to exist.

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Saddle Points

A differentiable function $f(x,y)$ has a **saddle point** at a critical point (a,b) if there are domain points (x,y) near (a,b) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface.

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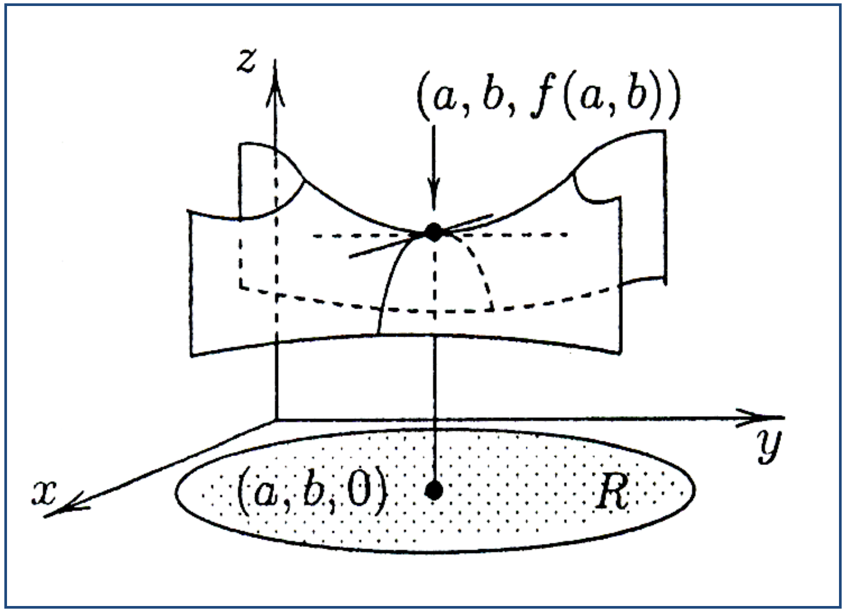

Point $(a,b,f(a,b))$ is a saddle point

LOCAL EXTREMA

Second Derivative Test

If $f(x,y)$ and its first and second derivatives are continuous near a critical point (a,b) then

- f has local maximum at (a,b) if $\Longrightarrow$ $f_{xx} f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} < 0$
- f has local minimum at (a,b) if $\Longrightarrow$ $f_{xx} f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} > 0$
- f has saddle point at (a,b) if $\Longrightarrow$ $f_{xx} f_{yy} - f_{xy}{}^2 < 0$

Saddle point at (a,b) *is* $(a,b,f(a,b))$

- Test fails at (a,b) if $\Longrightarrow$ $f_{xx} f_{yy} - f_{xy}{}^2 = 0$

$$\mathrm{D} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}{}^2$$

Example: Locate and identify the critical points of the following function

$$f(x,y)=\frac{y^3}{3}-x^2-y$$

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$$f(x,y)=\frac{y^3}{3}-x^2-y$$

Solution:

Step I: Find the critical points

$$f_x=-2x \quad , \quad f_y=y^2-1$$

$$f_x=-2x=0 \quad \Rightarrow \quad x=0$$

$$f_y=y^2-1=0 \quad \Rightarrow \quad y=\pm 1$$

Critical points are (0,1) and (0,-1)

Step II: Find the Second derivatives

$$f_{xx}=-2 \qquad f_{xy}=0 \qquad f_{yy}=2y$$

$$\mathrm{D}=f_{xx}f_{yy}-f_{xy}^2=-2(2y)-0=-4y$$

Step III: Test the critical points

At (0,1) $\quad D = -4y = -4(1) = -4 < 0$

$$f(0,1) = \frac{1}{3} - 0 - 1 = -\frac{2}{3}$$

$\left(0,1,-\frac{2}{3}\right)$ Is a saddle point

At (0,-1) $\quad D = -4(-1) = 4 > 0$

$$f_{xx}(0,-1) = -2 < 0$$

$$f(0,-1) = \frac{(-1)^3}{3} - 0 - 1 = \frac{2}{3}$$

$\left(0,\text{-}1,\frac{2}{3}\right)$ Is a maximum point

Example: Locate and identify the critical points of the following function

$$f(x, y) = 4xy - x^4 - y^4$$

Example: Locate and identify the critical points of the following function

$$f(x,y) = 4xy - x^4 - y^4$$

Solution:

Step I: Find the critical points

$$f_x = 4y - 4x^3 \quad , \quad f_y = 4x - 4y^3$$

$$f_x = 4y - 4x^3 = 0 \Rightarrow \quad y = x^3$$

$$f_y = 4x - 4y^3 = 0 \Rightarrow \quad x - x^9 = 0$$

$$x(1 - x^8) = 0 \Rightarrow x = 0, 1, -1$$

Critical points are (0,0), (1,1) and (-1,-1)

Step II: Find the Second derivatives

$$f_{xx} = -12x^2 \qquad f_{xy} = 4 \qquad f_{yy} = -12y^2$$

$$D = f_{xx} f_{yy} - f_{xy}^2 = -12x^2(-12y^2) - 16 = 144x^2y^2 - 16$$

Step III: Test the critical points

At (0,0) $\quad D = 144x^2y^2 - 16 = -16 < 0$

$f(0,0) = 0$

$(0, 0, 0)$ Is a saddle point

At (1,1) $\quad D = 144x^2y^2 - 16 = 128 > 0$

$f_{xx} = -12x^2 = -12 < 0$

$f(1,1) = 2$

$(1, 1, 2)$ Is a maximum point

At (-1,-1) $\quad D = 144x^2y^2 - 16 = 128 > 0$

$f_{xx} = -12x^2 = -12 < 0$

$f(-1,-1) = 2$

$(-1, -1, 2)$ Is a maximum point

Example: Locate and identify the critical points of the following function

$$f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$$

Example: Locate and identify the critical points of the following function

$$f(x,y)=\frac{1}{x}+xy+\frac{1}{y}$$

Solution:

Step I: Find the critical points

$$f_x=-1/x^2+\mathrm{y} \quad , \quad f_{\mathrm{y}}=x-1/y^2$$

$$f_x=-1/x^2+\mathrm{y}=0 \Rightarrow \quad y=1/x^2$$

$$f_y=x-1/y^2=0 \Rightarrow \quad x-x^4=0$$

$$x(1-x^3)=0 \Rightarrow x=0,1$$

Critical point is (1,1)

Note: The point (0,0) is not in the domain of f(x,y), so it can't be critical point.

Step II: Find the Second derivatives

$$f_{xx} = 2/x^3 \qquad f_{xy} = 1 \qquad f_{yy} = 2/y^3$$

$$D = f_{xx} f_{yy} - f_{xy}^2 = (2/x^3)(2/y^3) - 1 = 4/x^3y^3 - 1$$

Step III: Test the critical points

$$\text{At } (1,1) \qquad D = 4/x^3y^3 - 1 = 3 > 0$$

$$f_{xx} = 2/x^3 = 2 > 0$$

$$f(1,1) = 2$$

$(1,1,2)$ Is a minimum point

ABSOLUTE EXTREMA

Example: Find the absolute extremum of the function

$$f(x,y) = x^2 + xy + y^2 - 6x + 2$$

defined in the region R: $R = \{(x,y) : 0 \le x \le 5, -3 \le y \le 0\}$

Solution:

Example: Find the absolute extremum of the function

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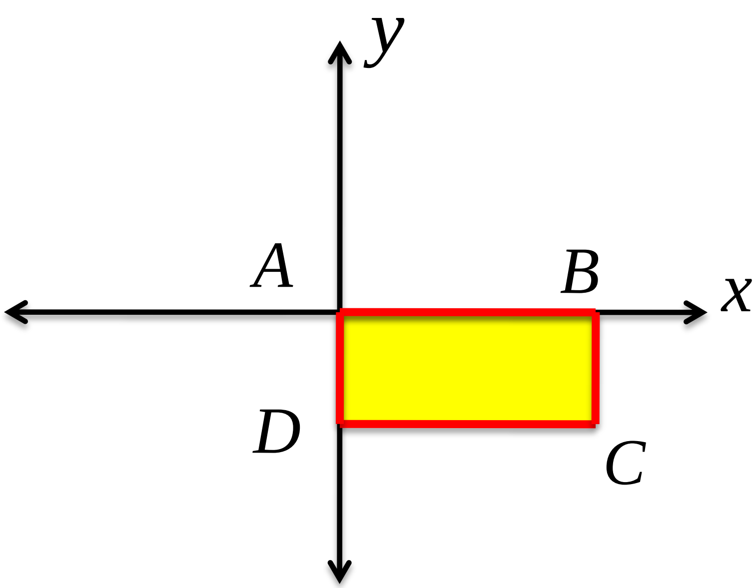

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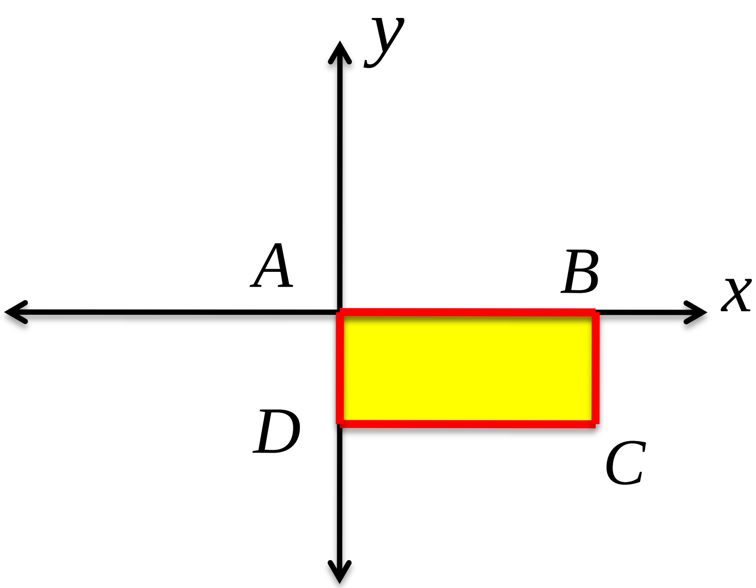
Solution:

Interior points

$$f_x = 2x + y - 6 \quad , \quad f_y = 2y + x$$

$$f_y = 2y + x = 0 \Rightarrow \quad x = -2y$$

$$f_x = 2x + y - 6 = 0 \Rightarrow \quad -3y - 6 = 0$$



Critical point is (4,-2)

$$f(4,-2) = (4)^2 + 4(-2) + (-2)^2 - 6(4) + 2 = -10$$

Boundary points

1: Along AB: y=0

$$f(x,0) = x^2 - 6x + 2$$

$$\frac{df}{dx} = 2x - 6 = 0 \Rightarrow x = 3$$

Critical point on AB is $(3,0)$

$$f(3,0) = (3)^2 - 6(3) + 2 = -7$$

End points of AB

$$f(0,0) = (0)^2 - 6(0) + 2 = 2$$

$$f(5,0) = (5)^2 - 6(5) + 2 = -3$$

2: Along BC; x=5

$$f(5, y) = y^2 + 5y - 3$$

$$\frac{df}{dy} = 2y + 5 = 0 \Rightarrow y = -5/2$$

Critical point on BC is (5,-5/2)

$$f(5,-5/2) = (-5/2)^2 + 5(-5/2) - 3 = -37/4$$

End points of BC

$$f(5,0) = (0)^2 + 5(0) - 3 = -3$$

$$f(5,-3) = (-3)^2 + 5(-3) - 3 = -9$$

3: Along CD; y=-3

$$f(x,-3) = x^2 - 9x + 11$$

$$\frac{df}{dx} = 2x - 9 = 0 \Rightarrow x = 9/2$$

Critical point on AC is (9/2,-3)

$$f(9/2,-3) = (9/2)^2 - 9(9/2) + 11 = -37/4$$

End points of CD

$$f(5,-3) = (5)^2 - 9(5) + 11 = -9$$

$$f(0,-3) = (0)^2 - 9(0) + 11 = 11$$

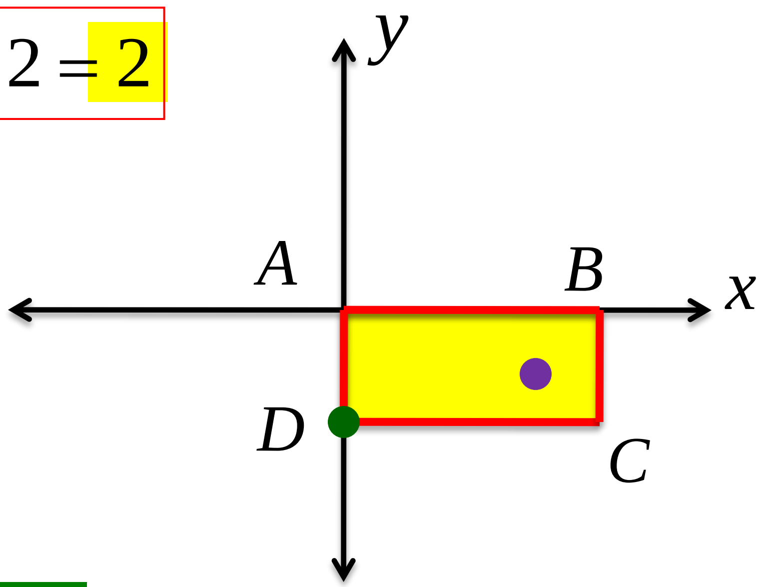
2: Along AD; x=0

$$f(0,y) = y^2 + 2$$

$$\frac{df}{dy} = 2y = 0 \Rightarrow y = 0$$

Critical point on AD is (0,0)

$$f(0,0) = 0^2 + 2 = 2$$



End points of AD

$$f(0,-3) = (-3)^2 + 2 = 11$$

$$f(0,0) = 0^2 + 2 = 2$$

Absolute Maximum $f(0,-3) = 11$

Absolute Minimum $f(4,-2) = -10$

Example: Find the absolute extremum of the function

$$f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$$

defined in the region R bounded by lines x = 0, y = 2 and y = 2x.

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$$f(x,y)=2x^2-4x+y^2-4y+1$$

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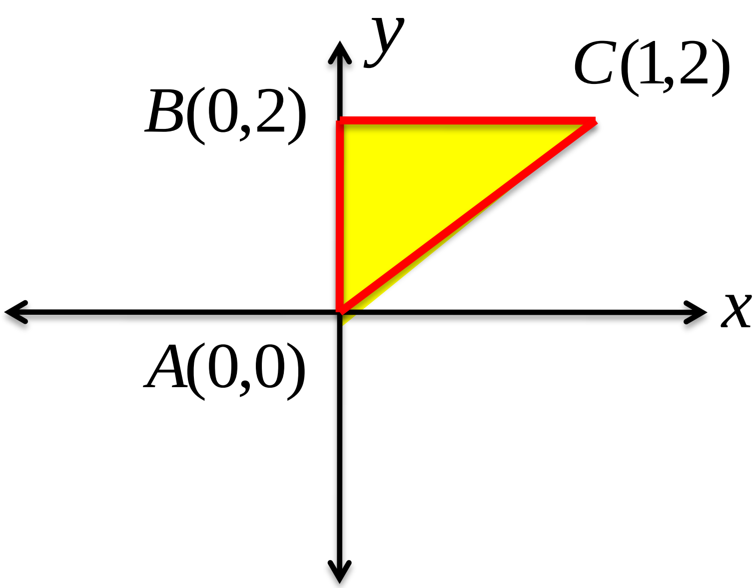
Solution:

Interior points

$$f_x=4x-4 \quad , \quad f_y=2y-4$$

$$f_x=4x-4=0 \Rightarrow \quad x=1$$

$$f_y=2y-4=0 \Rightarrow \quad y=2$$



Critical point is (1,2) BUT (1,2) IS NOT AN INTERIOR POINT

Boundary points

1: Along AB; x=0

$$f(0,y) = y^2 - 4y + 1$$

$$\frac{df}{dy} = 2y - 4 = 0 \Rightarrow y = 2$$

Critical point on AB is (o,2)

$$f(0,2) = (2)^2 - 4(2) + 1 = -3$$

End points of AB

$$f(0,0) = (0)^2 - 4(0) + 1 = 1$$

$$f(0,2) = (2)^2 - 4(2) + 1 = -3$$

2: Along BC; y=2

$$f(x,2) = 2x^2 - 4x + -3$$

$$\frac{df}{dx} = 4x - 4 = 0 \Rightarrow x = 1$$

Critical point on BC is (1,2)

$$f(1,2) = 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = -5$$

End points of BC

$$f(0,2) = 2(0)^2 - 4(0) - 3 = -3$$

$$f(1,2) = 2(1)^2 - 4(1) - 3 = -5$$

3: Along AC; y=2x

$$f(x,2x) = 6x^2 - 12x + 1$$

$$\frac{df}{dx} = 12x - 12 = 0 \Rightarrow x = 1$$

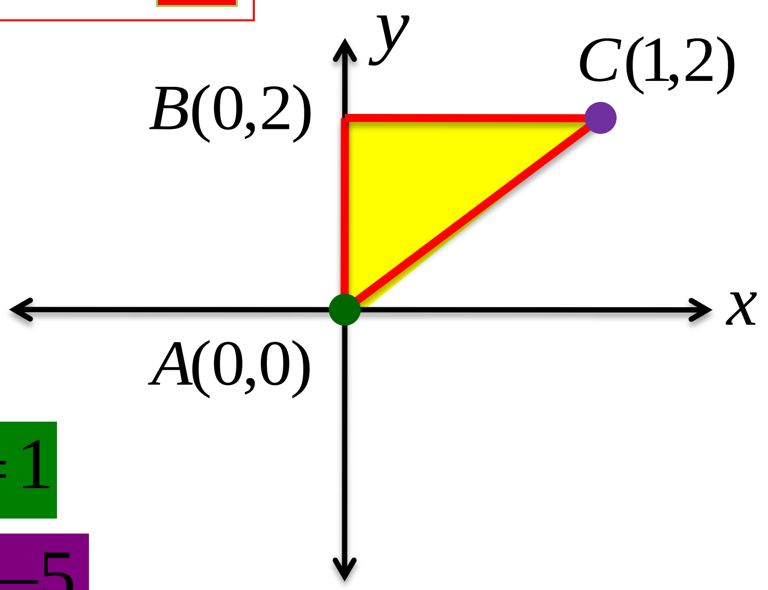
Critical point on AC is $(1,2)$

$$f(1,2) = 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = -5$$

End points of AC

$$f(0,0) = 6(0)^2 - 12(0) + 1 = 1$$

$$f(1,2) = 2(1)^2 - 4(1) - 3 = -5$$



Absolute Maximum $f(0,0) = 1$

Absolute Minimum $f(1,2) = -5$

Example: Find the absolute extremum of the function

$$f(x,y) = x^2 + xy - 2x - y/2$$

defined in the region R bounded by lines $y = x^2$ and $y = 2 - x^2$.

Solution:

Example: Find the absolute extremum of the function

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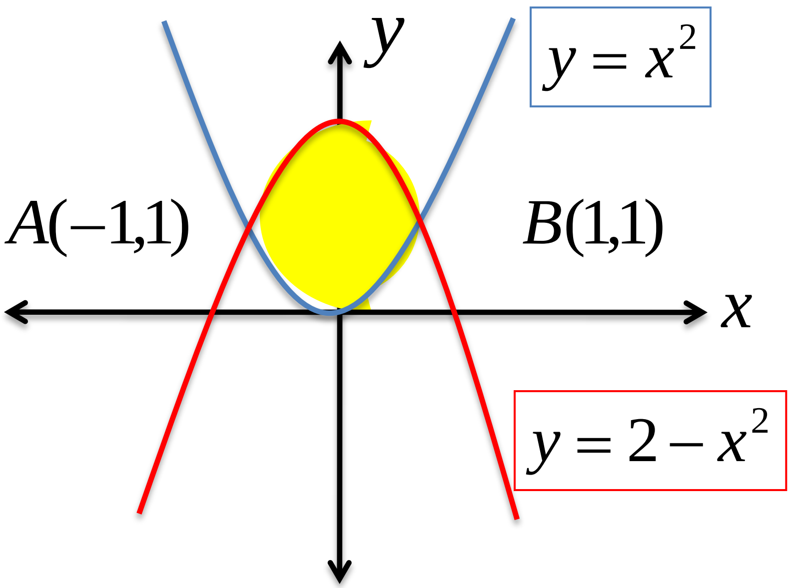

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defined in the region R bounded by lines $y = x^2$ and $y = 2 - x^2$.

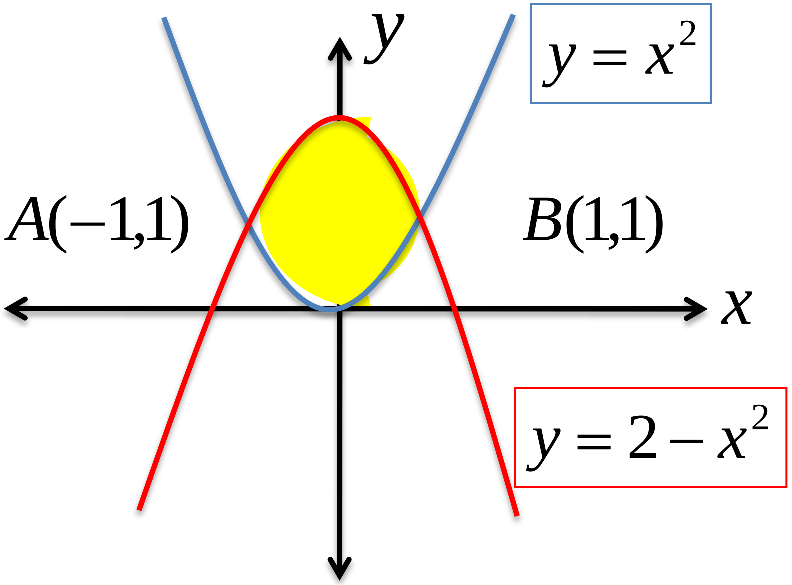
Solution:

Interior points

$$f_x = 2x + y - 2 \quad , \quad f_y = x - 1/2$$

$$f_y = x - 1/2 = 0 \Rightarrow \quad x = 1/2$$

$$f_x = 2x + y - 2 = 0 \Rightarrow \quad y = 1$$



Critical point is (1/2,1)

$$f(1/2,1) = (1/2)^2 + (1/2)1 - 2(1/2) - 1/2 = -3/4$$

Boundary points

1: Along $y = x^2$

$$f(x, x^2) = x^3 + x^2/2 - 2x$$

$$\frac{df}{dx} = 3x^2 + x - 2 = 0 \Rightarrow x = -1, 2/3$$

Critical point on $y = x^2$ are (-1,1) and (2/3,4/9)

$$f(-1,1) = (-1)^3 + (-1)^2/2 - 2(-1) = 3/2$$

$$f(2/3,4/9) = (2/3)^3 + (2/3)^2/2 - 2(2/3) = -22/27$$

2: Along $y = 2 - x^2$

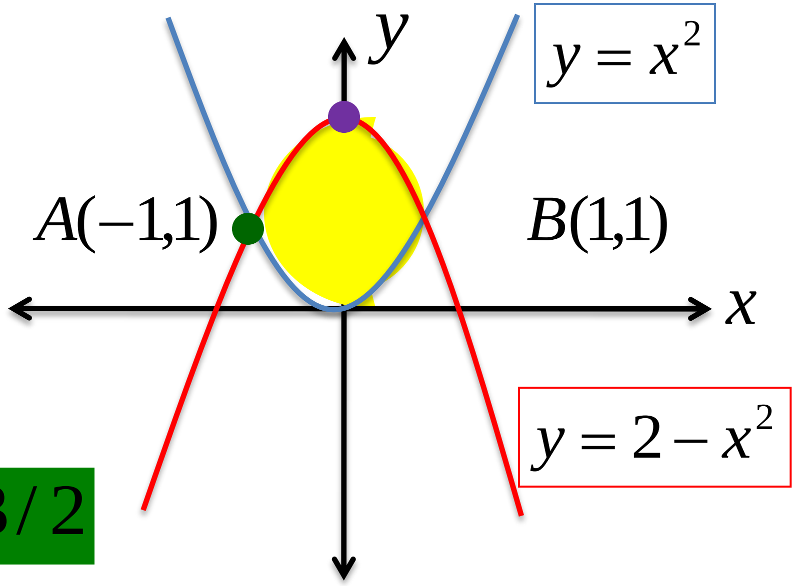
$$f(x, 2 - x^2) = -x^3 + 3x^2/2 - 1$$

$$\frac{df}{dx} = -3x^2 + 3x = 0 \Rightarrow x = 0, 1$$

Critical point on $y = 2 - x^2$ are (0,2) and (1,1)

$$f(0,2) = -(0)^3 + 3(0)^2/2 - 1 = -1$$

$$f(1,1) = -(1)^3 + 3(1)^2/2 - 1 = -1/2$$



Absolute Maximum $f(-1,1) = 3/2$

Absolute Minimum $f(0,2) = -1$