Double Integrals

Integration

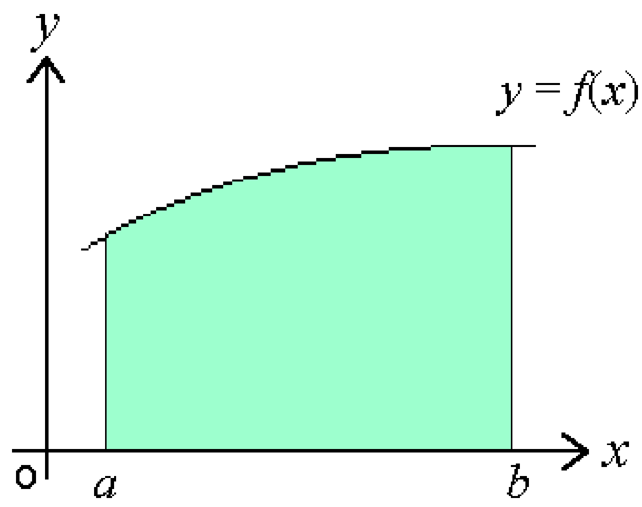
It is an *underestimation* to say that integration is an anti-derivative or an operation that gives area under the curve. This is true but do not reflect the main purpose of integration nor does it indicate the diversity in it. The original concept of integration in simple words can be stated in the following way.

“Suppose that there is a quantity which can be divided into infinitely many small pieces such that if we join all of them the original quantity is attained.”

The concept of integration helps us to find that quantity by adding together small pieces. ***The process of continuously adding together all small pieces is known as integration***.

Area under curve

We can find an expression for the area A bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$.



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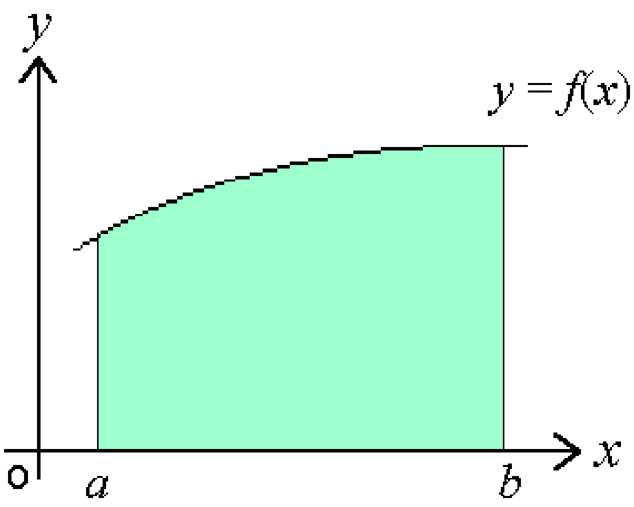
$$A = \int_a^b f(x)dx$$



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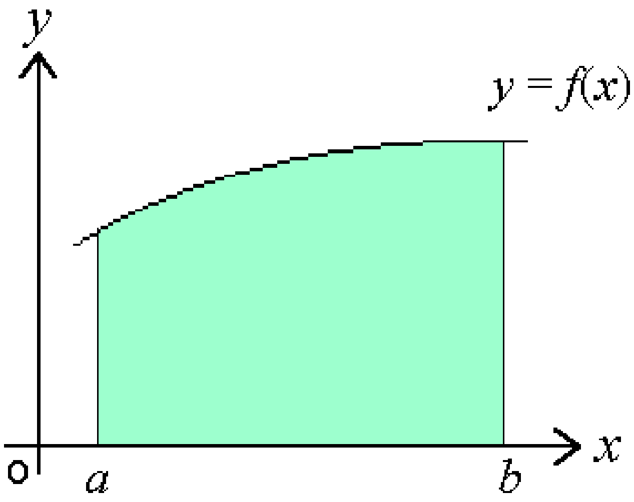

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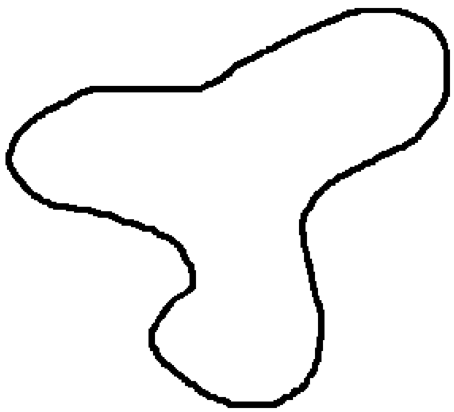

Can you find area of this irregular shape shown?

Area between curves

Divide the irregular shape into parts such that you can measure area of each part by using $A = \int_a^b f(x)dx$

$$A = A_1 + A_2$$

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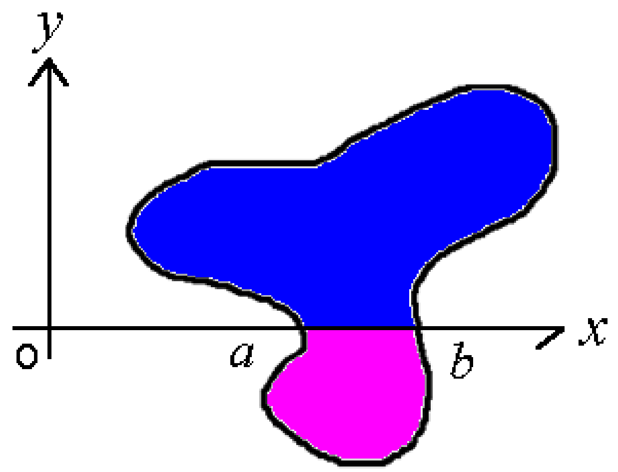

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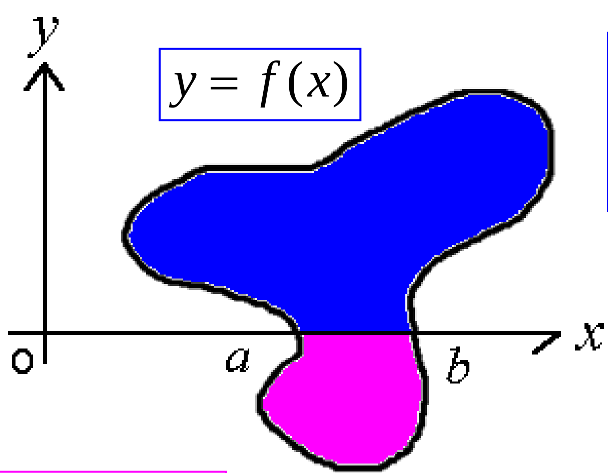

$$A_1 = \int_a^b f(x)dx$$

$$A_2 = \int_a^b g(x)dx$$

$y = g(x)$

Double Integration

Cartesian Coordinates

Double Integrals

The graph of function of two variables f(x, y) is surface, plotted in 3-D. The integral of f(x, y) over the region R is the volume between the graph and the region R.

$$V = \iint_R f(x, y)\,dA$$

where dA is dxdy or dydx

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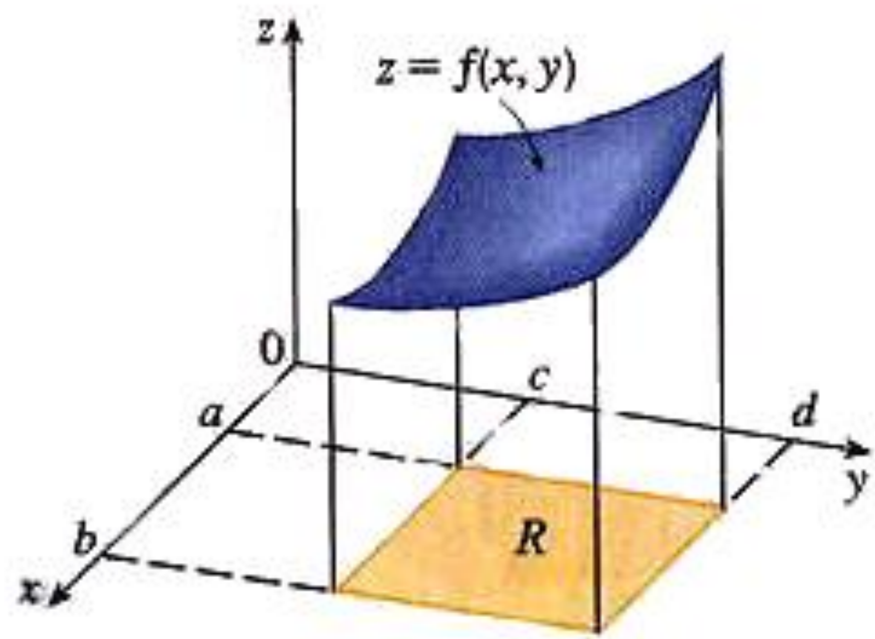

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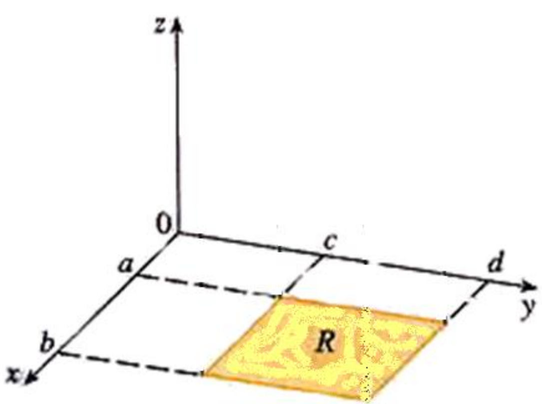
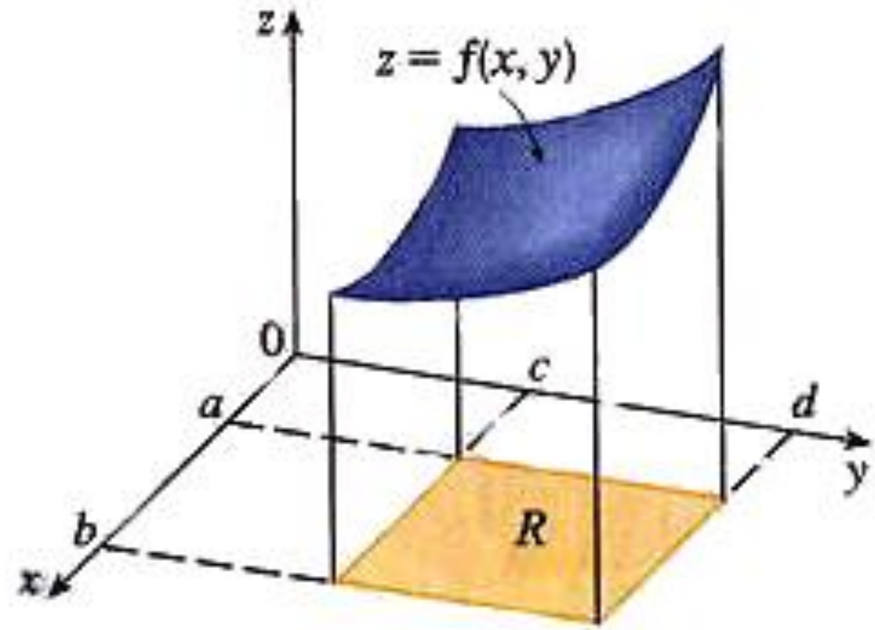
$$A = \iint_R dA$$



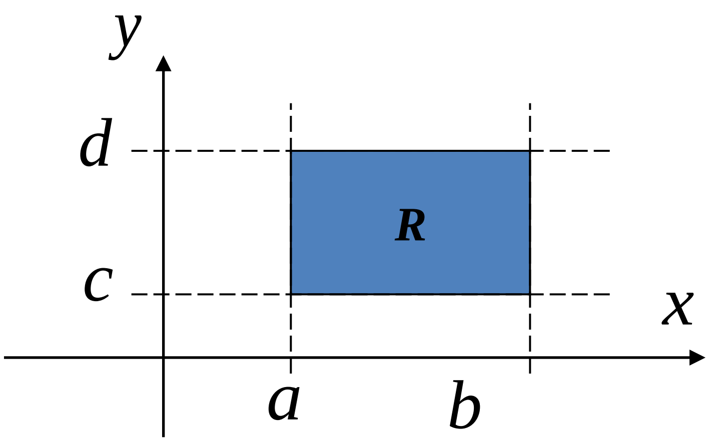


Double Integrals

If $f(x,y)$ is continuous in a rectangula r region

$$R = \{(x, y) : a \le x \le b, c \le y \le d\}$$

then



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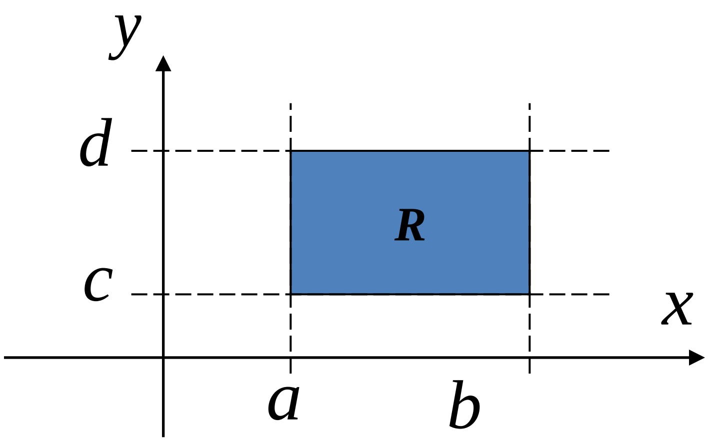

$$\int_c^d \int_a^b f(x, y)dxdy = \int_c^d \left[\int_a^b f(x, y)dx \right] dy$$

$$\int_a^b \int_c^d f(x, y)dxdy = \int_a^b \left[\int_c^d f(x, y)dy \right] dx$$

Double Integrals

Practice: Evaluated the iterated integrals (i) $\int_{1}^{2}\int_{-1}^{1}(3x^2+2xy)\,dy\,dx$ (ii) $\int_{0}^{1}\int_{x^2}^{x}30y\,dy\,dx$

Double Integrals

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Sol:

$$\int_1^2\int_{-1}^1 (3x^2+2xy)\,dy\,dx = \int_1^2 (3x^2y+xy^2)_{-1}^1\,dx$$

$$= \int_1^2 \left\{[3x^2(1)+x(1)^2]-[3x^2(-1)+x(-1)^2]\right\}dx$$

$$= \int_1^2 6x^2\,dx = (2x^3)_1^2 = 2(2)^3-2(1)^3 = 14$$

Double Integrals

Practice: Evaluated the iterated integrals (i) $\int_{1}^{2}\int_{-1}^{1}(3x^2+2xy)dydx$ (ii) $\int_{0}^{1}\int_{x^2}^{x}30ydydx$

Sol:

$$\int_{1}^{2}\int_{-1}^{1}(3x^2+2xy)dydx=\int_{1}^{2}(3x^2y+xy^2)_{-1}^{1}dx$$

$$=\int_{1}^{2}\left\{[3x^2(1)+x(1)^2]-[3x^2(-1)+x(-1)^2]\right\}dx$$

$$=\int_{1}^{2}6x^2dx=(2x^3)_1^2=2(2)^3-2(1)^3=14$$

Sol:

$$\int_{0}^{1}\int_{x}^{x^2}30ydydx=\int_{0}^{1}(15y^2)_{x^2}^{x}dx$$

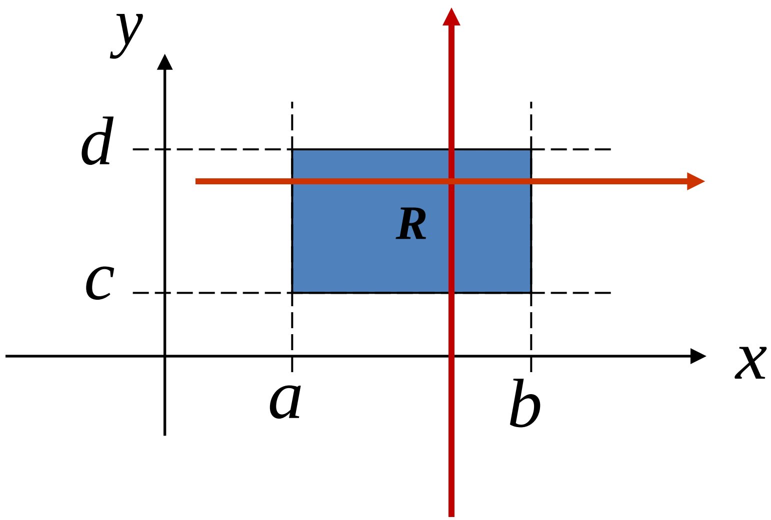
$$=\int_{0}^{1}(15x^2-15x^4)dx$$

$$=(5x^3-3x^5)_0^1=5-3=2$$

Fubini's theorem (Rectangular Region)

If f(x, y) is continuous on the region R

$$R = \{(x, y) : a \le x \le b, c \le y \le d\}$$



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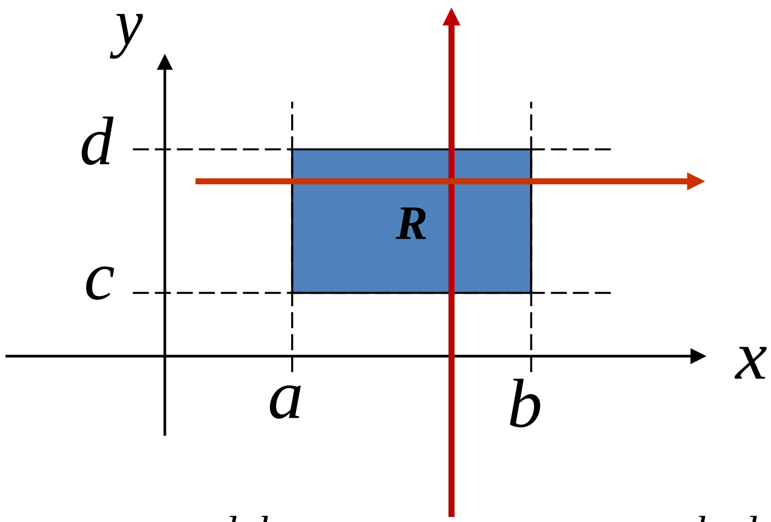


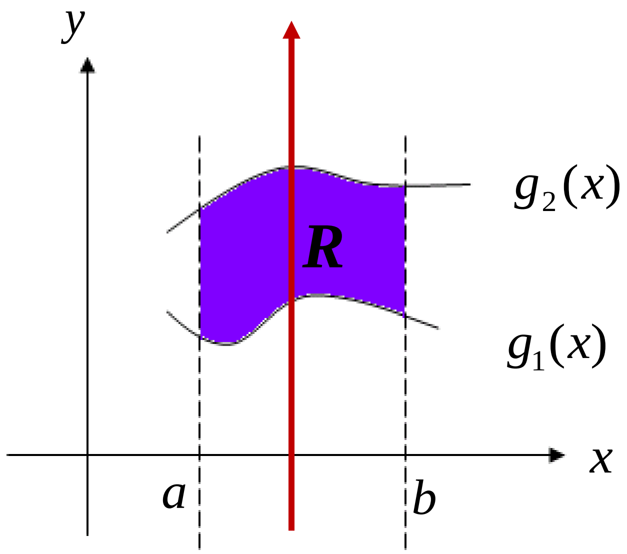
$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Fubini's theorem

Type I

If f(x, y) is continuous on the region R

$$R = \{(x, y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$$



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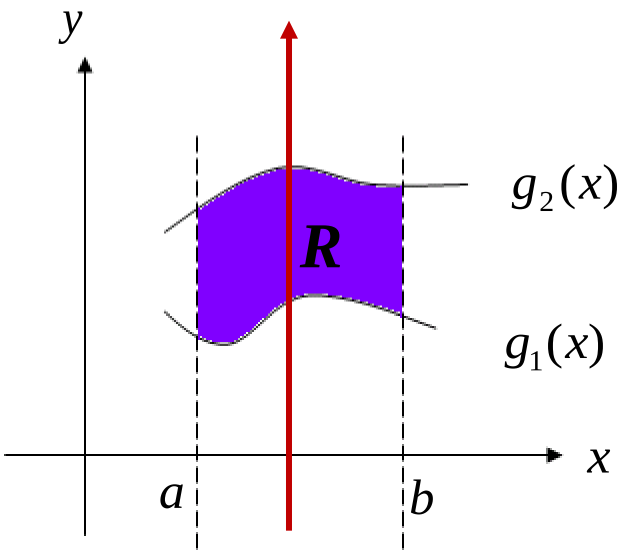

then

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

Fubini's theorem

Type I

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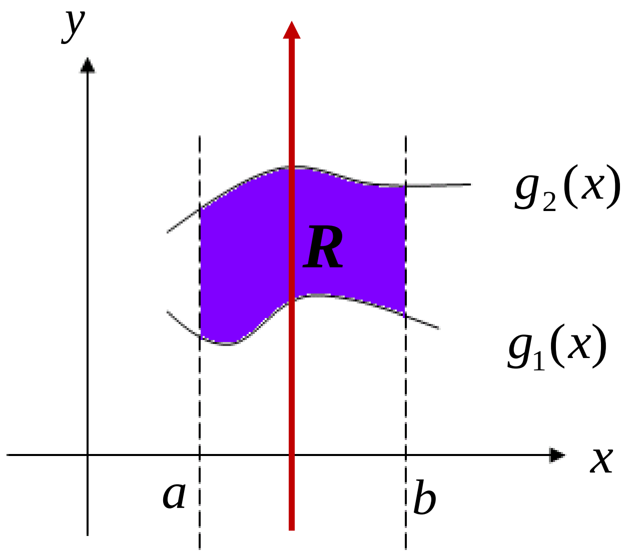
$$R = \{(x, y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$$



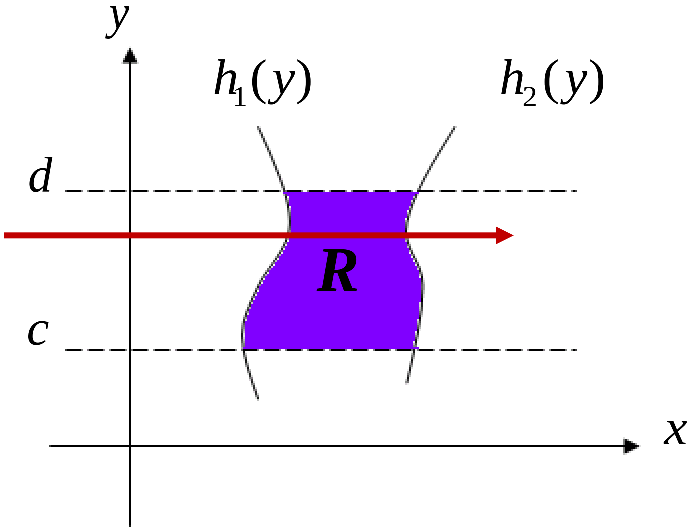
then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II

If f(x, y) is continuous on the region R

$$R = \{(x, y): c \le y \le d, h_1(y) \le x \le h_2(y)\}$$



then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Double Integrals

Practice: Sketch the region of integration and evaluate $\iint_R f(x,y)\,dA$ where

$$f(x,y) = 4x - y; R = \{(x,y) : y^2 \le x \le 2y, 0 \le y \le 2\}$$

Sol:

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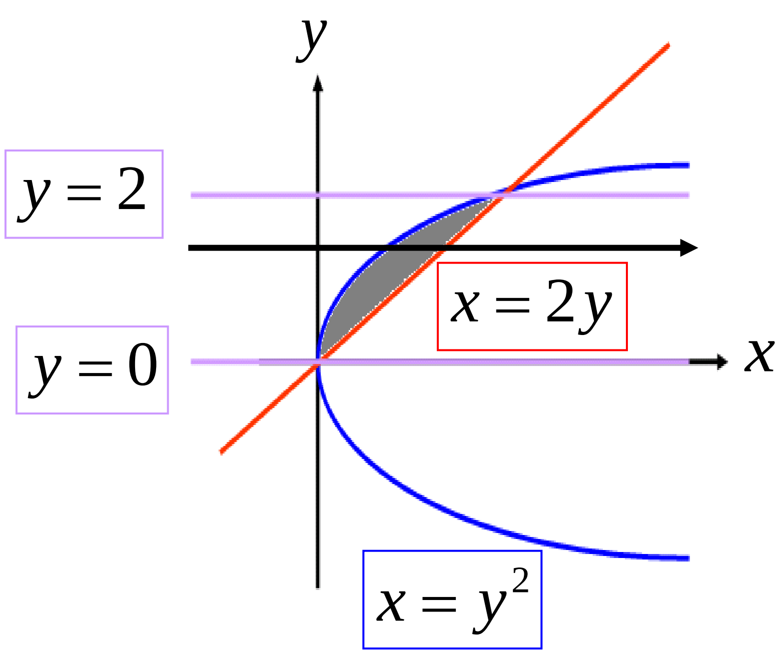
Sol:

The region R is



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$$f(x,y) = 4x - y;\ R = \{(x,y) : y^2 \le x \le 2y, 0 \le y \le 2\}$$

Sol:

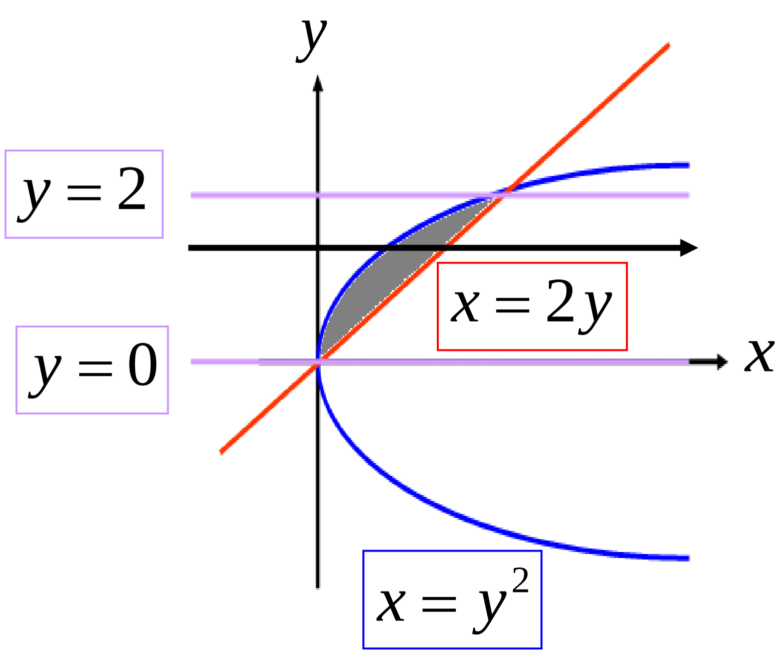
$$\iint_R (4x-y)\,dA = \int_0^2 \int_{y^2}^{2y} (4x-y)\,dx\,dy = \int_0^2 \left[2x^2 - xy\right]_{y^2}^{2y} dy$$

$$= \int_0^2 \{[2(2y)^2 - 2y^2] - [2(y^2)^2 - y^3]\}\,dy$$

$$= \int_0^2 (6y^2 + y^3 - 2y^4)\,dy$$

$$= \left[2y^3 + \frac{y^4}{4} - \frac{2y^5}{5}\right]_0^2 = \frac{36}{5}$$

The region R is



Double Integrals

Practice: Sketch the region of integratio n and evaluate $\iint_R f(x,y)\,dA$ where

$f(x,y) = y$; R is the region bounded $x = 0, x = \pi, y = 0$ and $y = \sin x$

Sol:

Double Integrals

Practice: Sketch the region of integratio n and evaluate $\iint_R f(x,y)\,dA$ where

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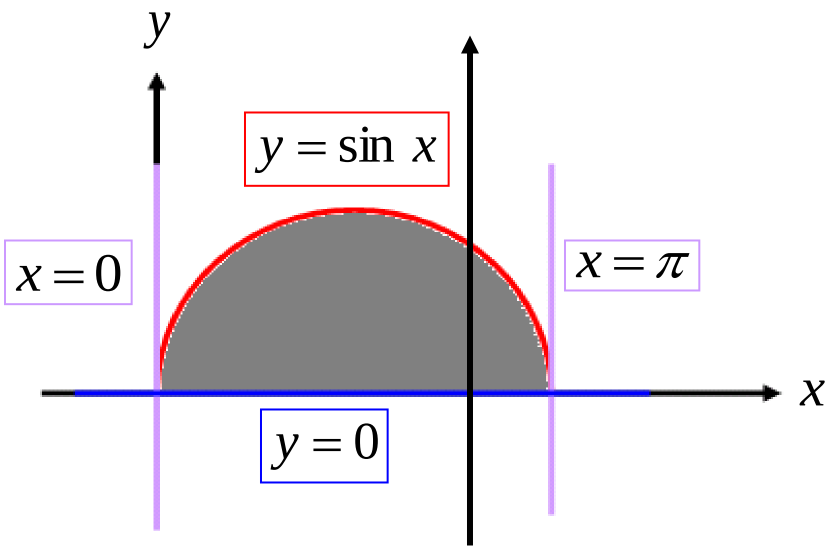

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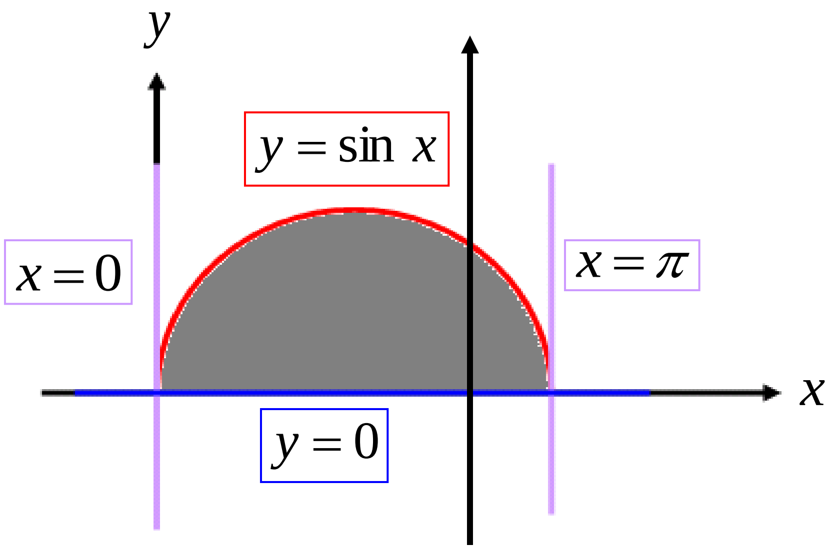
Sol:

The region R is



$$\iint_R y\,dA=\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx=\int_0^{\pi}\left(\frac{y^2}{2}\right)_0^{\sin x}dx$$

$$=\int_0^{\pi}\frac{\sin^2 x}{2}\,dx$$

$$=\frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$=\frac{1}{4}\left(x-\frac{\sin 2x}{2}\right)_0^{\pi}=\frac{\pi}{4}$$

Double Integrals

Practice: Sketch the region of integration and evaluate $\iint_R f(x,y)\,dA$ where $f(x,y) = xy^2$ and R is the closed triangular region with vertices (0,0), (3,1) and (-2,1)

Sol:

Double Integrals

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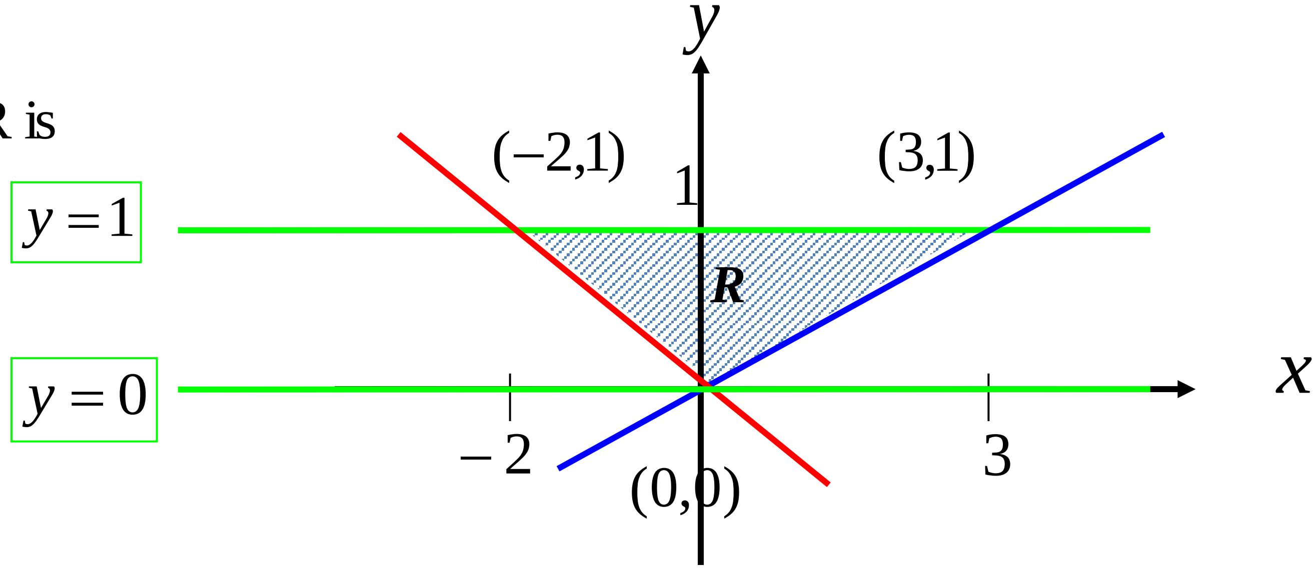

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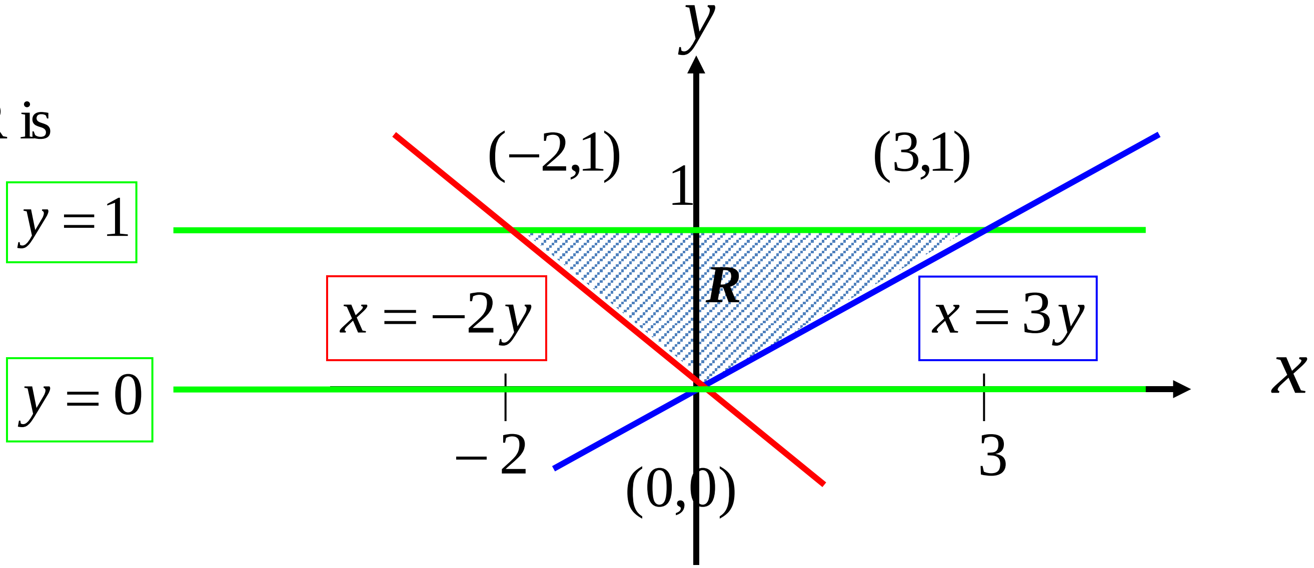

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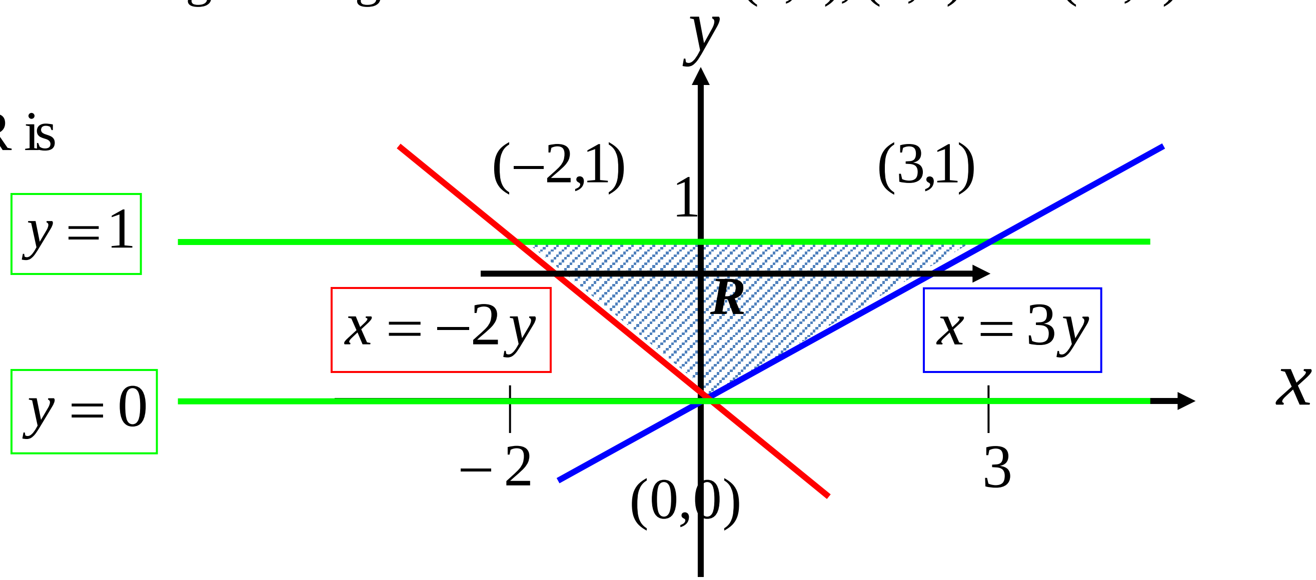
Sol: The region R is



$$\iint_R xy^2\,dA = \int_0^1 \int_{-2y}^{3y} xy^2\,dx\,dy = \int_0^1 \left[\frac{x^2y^2}{2}\right]_{-2y}^{3y} dy$$

$$= \int_0^1 \frac{y^2}{2}(9y^2 - 4y^2)\,dy = \int_0^1 \frac{5y^4}{2}\,dy = \left[\frac{y^5}{2}\right]_0^1 = \frac{1}{2}$$

Double Integrals

Practice: Sketch the region of integration, determine the order of integration and evaluate the following integrals

(i) $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$ and (ii) $\int_0^{2}\int_{y/2}^{1} e^{x^2}\, dx\, dy$

Sol:

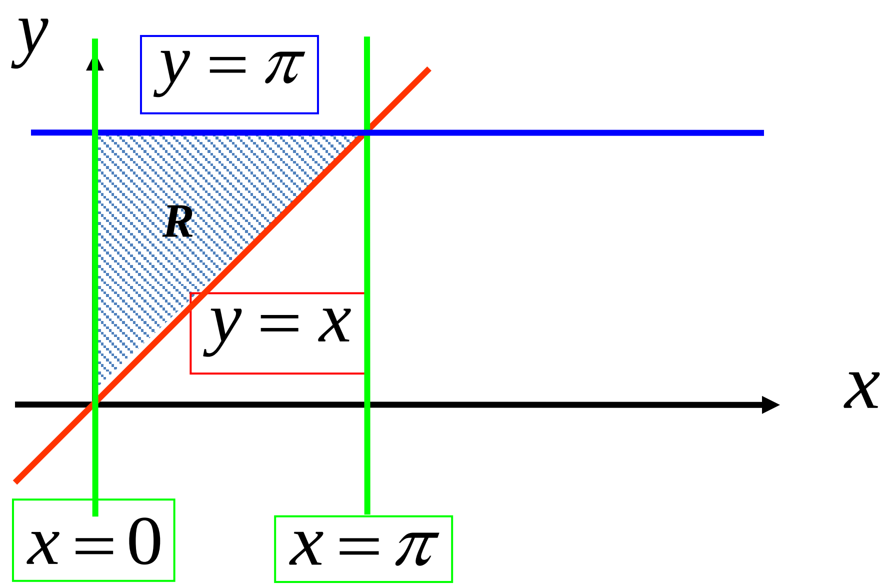
Double Integrals

Practice: Sketch the region of integratio n, determine the order of integratio n and evaluate the following integrals

(i) $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$ and (ii) $\int_0^{2}\int_{y/2}^{1} e^{x^2}\,dx\,dy$

Sol:

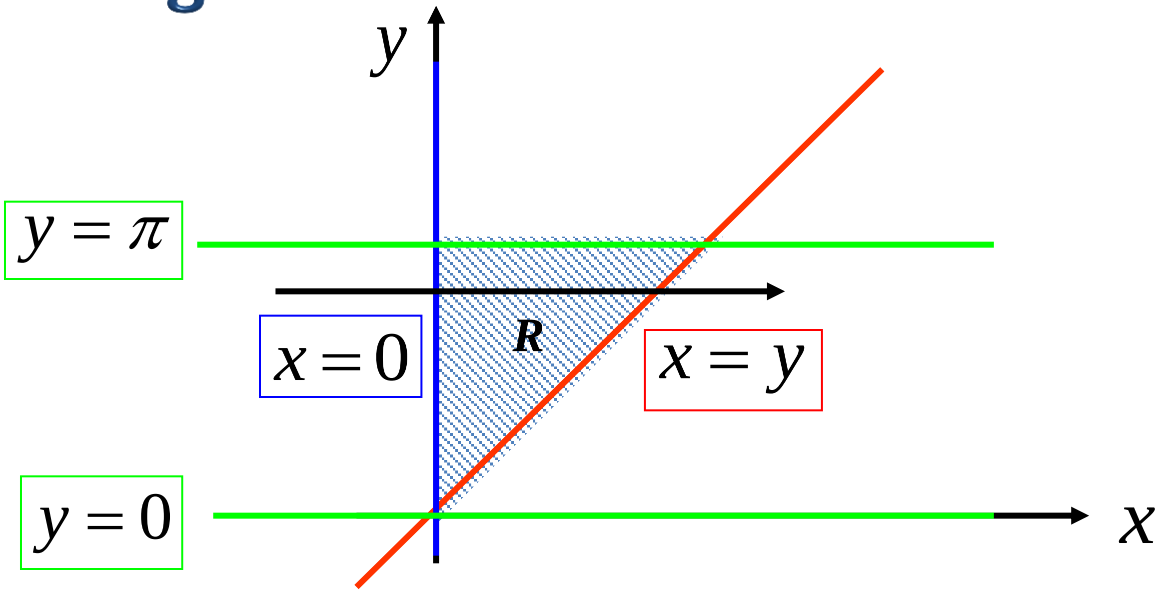
The region R is



But the inner integral $\int \frac{\sin y}{y}$ is difficult to integrate.

Why not to go other way round?

Double Integrals



$$\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx=\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}\,dx\,dy$$

$$=\int_0^{\pi}\frac{\sin y}{y}(x)_0^{y}\,dy=\int_0^{\pi}\frac{\sin y}{y}(y)\,dy$$

$$=\int_0^{\pi}\sin y\,dy=(-\cos y)_0^{\pi}=-\cos\pi+1=2$$

Double Integrals

Sol:

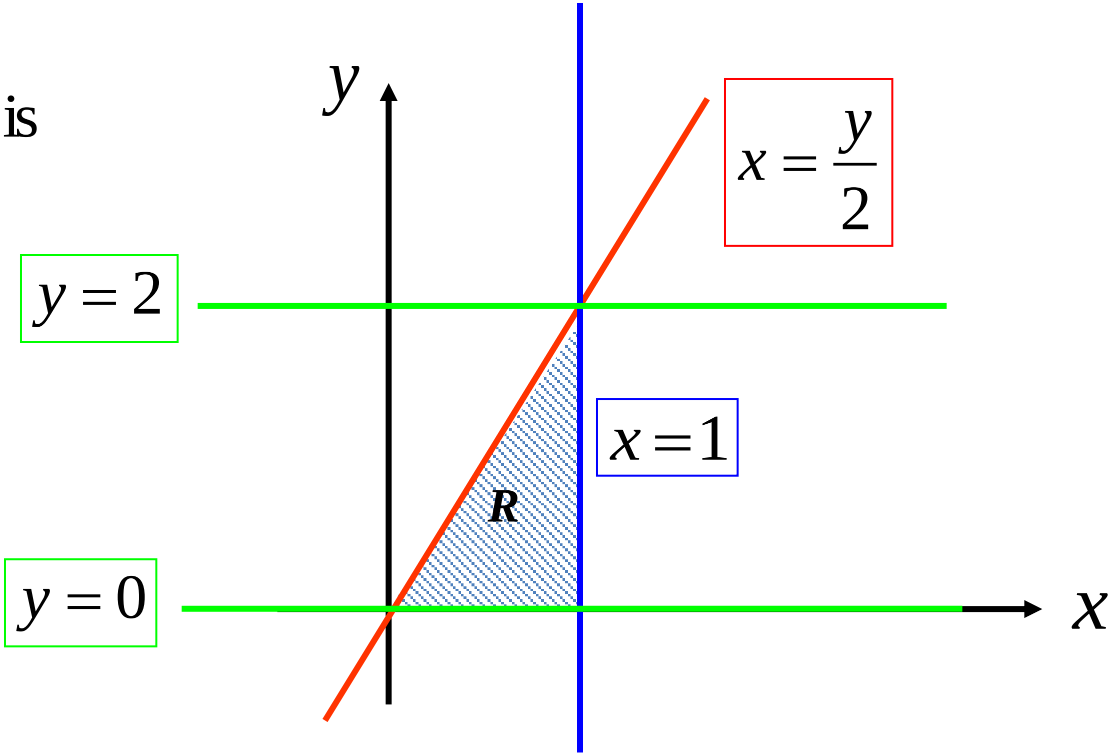
$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

Double Integrals

Sol:

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$$

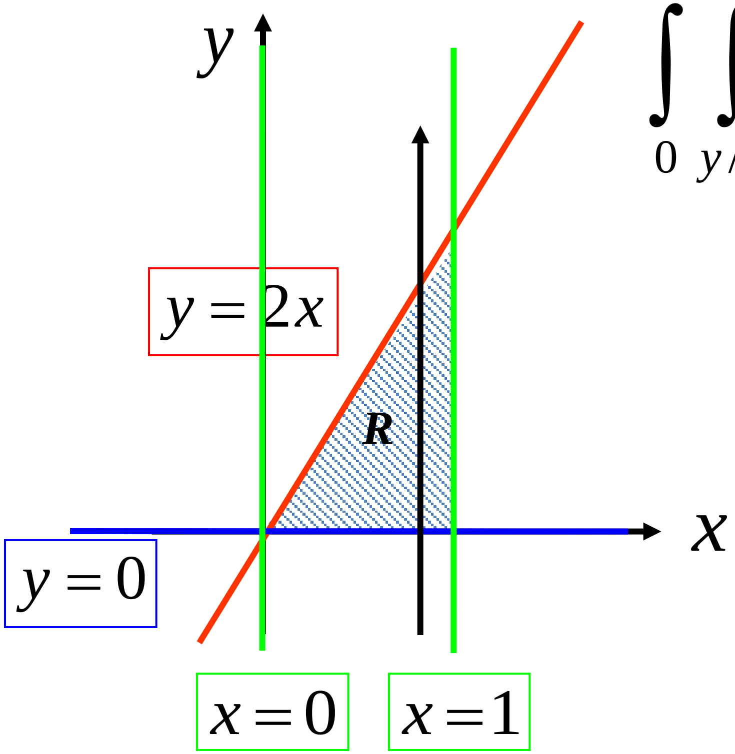
The region R is



But the inner integral $\int e^{x^2}dx$ is difficult to integrate.

Again try the other way round.

Double Integrals



$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$$

$$= \int_{x=0}^1 e^{x^2} (y)_0^{2x}\,dx$$

$$= \int_0^1 2x e^{x^2}\,dx$$

$$= \int_0^1 e^u\,du = (e^u)_0^1 = e - 1$$

More Applications of Double Integrals

How to find areas of plane curves?

Double Integrals

Practice: Sketch the region of integration and find the area of region

$$R = \{(x, y) : y^2 \le x \le 2y, 0 \le y \le 2\}$$

Sol:

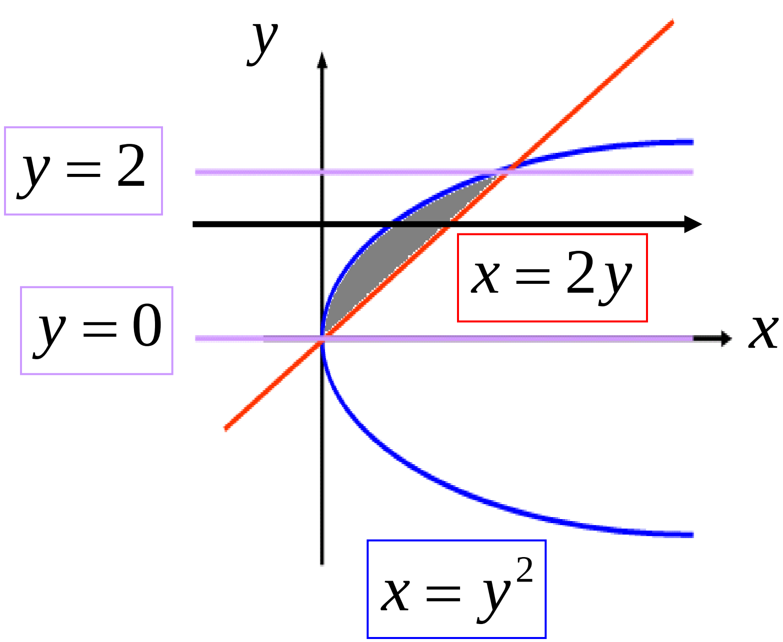
Double Integrals

Practice: Sketch the region of integration and find the area of region

$$R = \{(x, y) : y^2 \le x \le 2y, 0 \le y \le 2\}$$

Sol:

The region R is



$$A = \iint_R dA = \int_0^2 \int_{y^2}^{2y} dx\,dy = \int_0^2 (x)_{y^2}^{2y}\,dy$$

$$= \int_0^2 (2y - y^2)\,dy$$

$$= (y^2 - \frac{y^3}{3})_0^2 = \frac{4}{3}$$

Double Integrals

Practice: Sketch the region of integration and find the area of region R; region bounded $x = 0, x = \pi, y = 0$ and $y = \sin x$

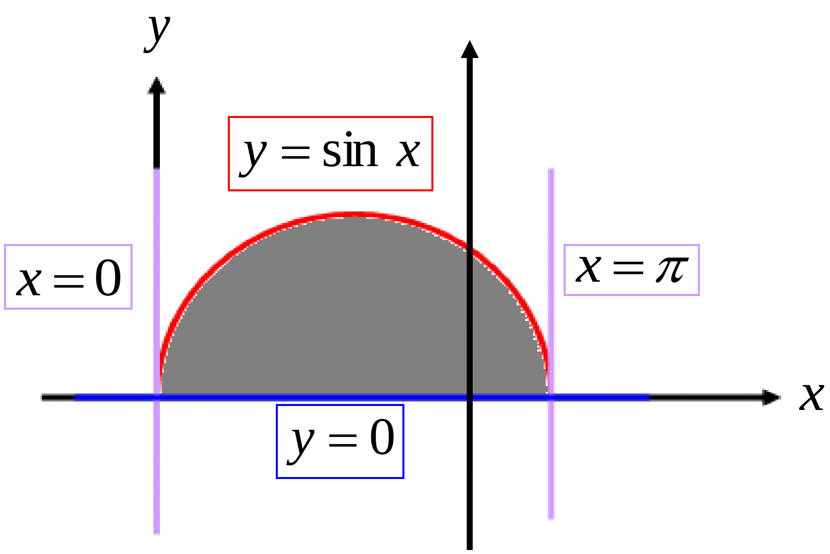
Sol:

Double Integrals

Practice: Sketch the region of integration and find the area of region R; region bounded $x=0, x=\pi, y=0$ and $y=\sin x$

Sol:

The region R is



$$A=\iint_R dA=\int_0^{\pi}\int_0^{\sin x} dy\,dx$$

$$=\int_0^{\pi}(y)_0^{\sin x}\,dx$$

$$=\int_0^{\pi}\sin x\,dx$$

$$=(-\cos x)_0^{\pi}=2 \quad ♣$$

Double Integrals

Practice: Sketch the region of integration and evaluate the area of region R : closed triangular region with vertices (0,0), (3,1) and (-2,1)

Sol:

Double Integrals

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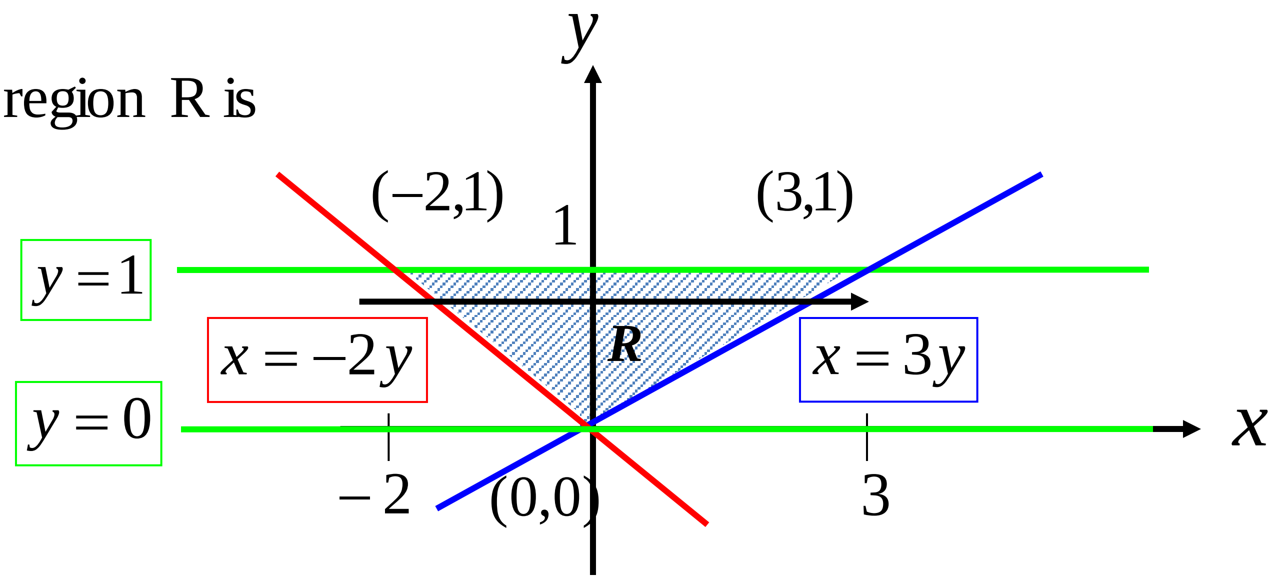
Sol:

The region R is



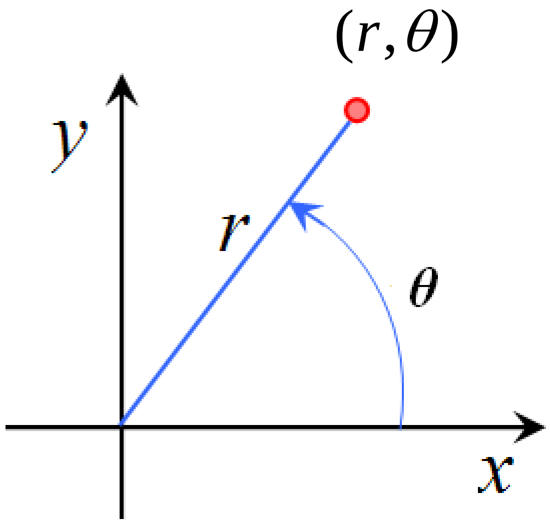
$$A=\iint_R dA=\int_0^1\int_{-2y}^{3y} dx\,dy$$

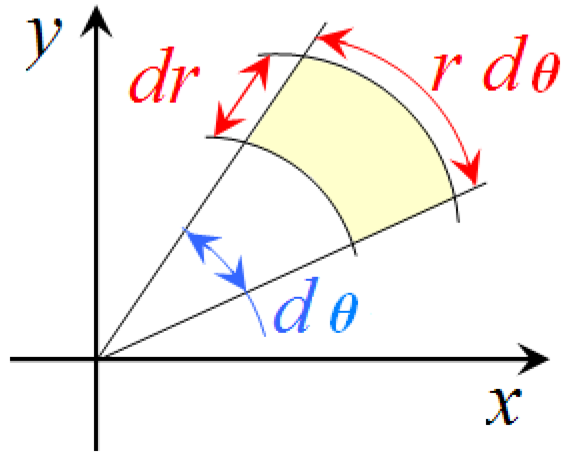
$$=\int_0^1 (x)_{-2y}^{3y}\,dy=\int_0^1(5y)\,dy=(5\frac{y^2}{2})_0^1=\frac{5}{2}$$

Areas of
Polar Curves

Polar Coordinates (r,θ)

$$0 \le r < \infty \quad 0 \le \theta < 2\pi$$





$$x = r\cos\theta \quad y = r\sin\theta$$

$$da = (dr)\times(rd\theta) = rdrd\theta$$

$$r = \sqrt{x^2 + y^2} \quad y/x = \tan\theta$$

Practice: Evaluate the following integral by changing to polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(x^2 + y^2)\, dy\, dx$$

Sol:

Difficulties:

1. Implicit functions
2. Integration will include inverse sin or cosine functions
3. Geometric understanding

Practice: Evaluate the following integral by changing to polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx$$

Sol:

Difficulties:

1. Implicit functions
2. Integration will include inverse sin or cosine functions
3. Geometric understanding

Solution

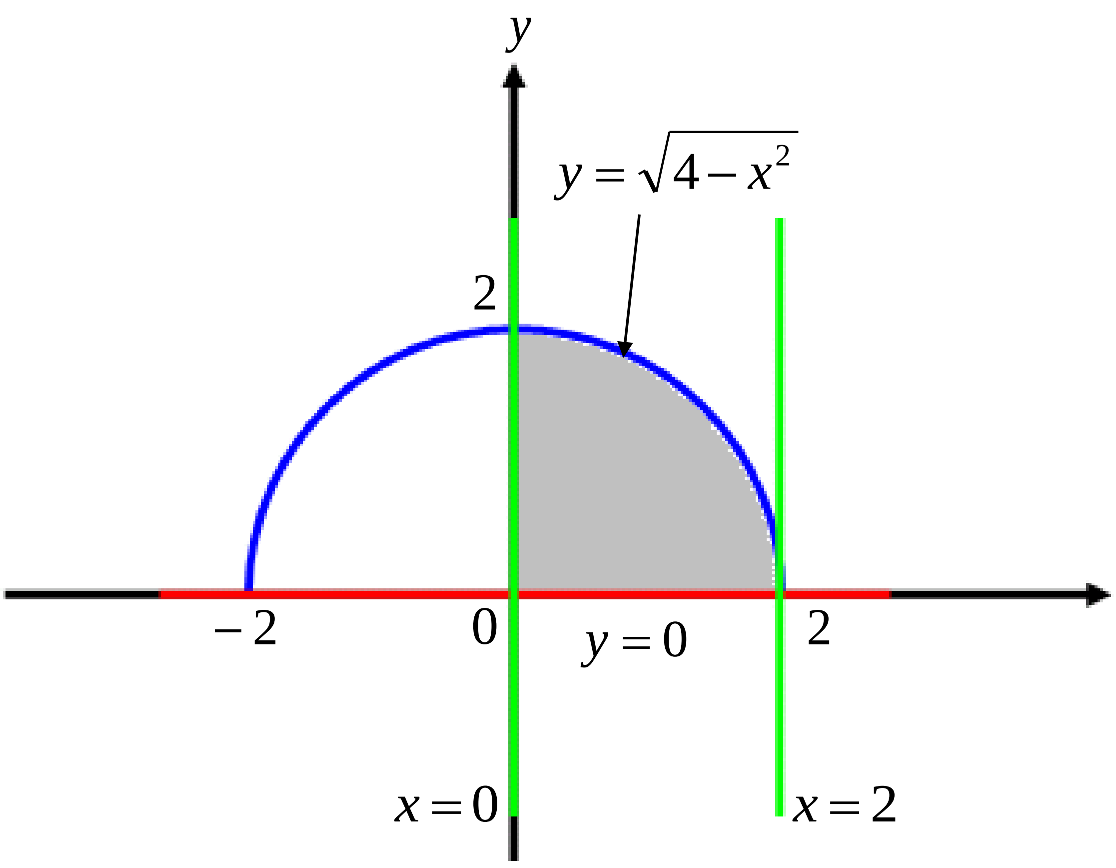
Go to the circular polar coordinates to deal with integration of quantities like x^2+y^2

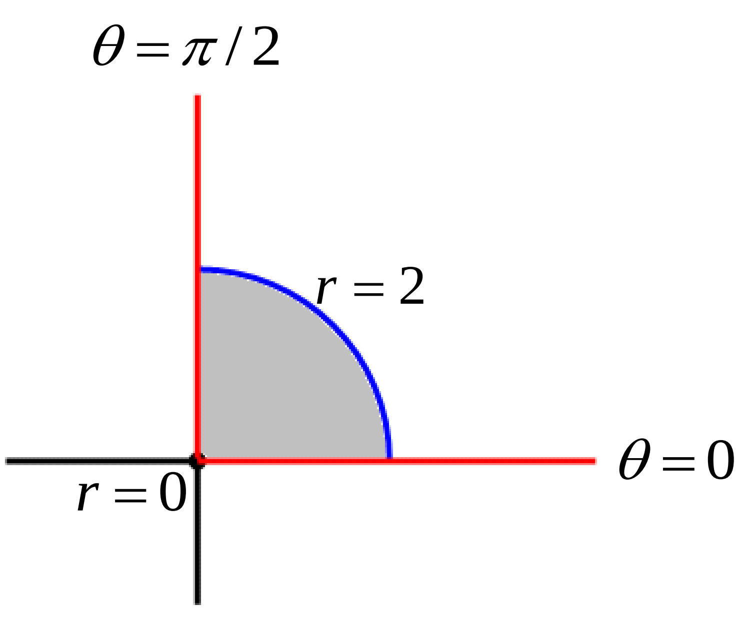
Practice: Evaluate the following integral by changing to polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx$$

Sol:

The region of integration R is described as





Therefore

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(x^2 + y^2)\,dy\,dx = \int_0^{\pi/2} \int_0^2 (\cos r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^4 \tfrac{1}{2}(\cos u)\,du\,d\theta$$

$$= \int_0^{\pi/2} \tfrac{1}{2}\left[\sin u\right]_0^4 d\theta =$$

$$= \int_0^{\pi/2} \tfrac{1}{2}\sin 4\,d\theta = \frac{\pi \sin 4}{4} \quad ♣$$

Practice: Evaluate the following integrals by changing to polar coordinates

$$\int_0^1 \int_0^y \left(\frac{y^2}{x^2 + y^2}\right) dx\,dy$$

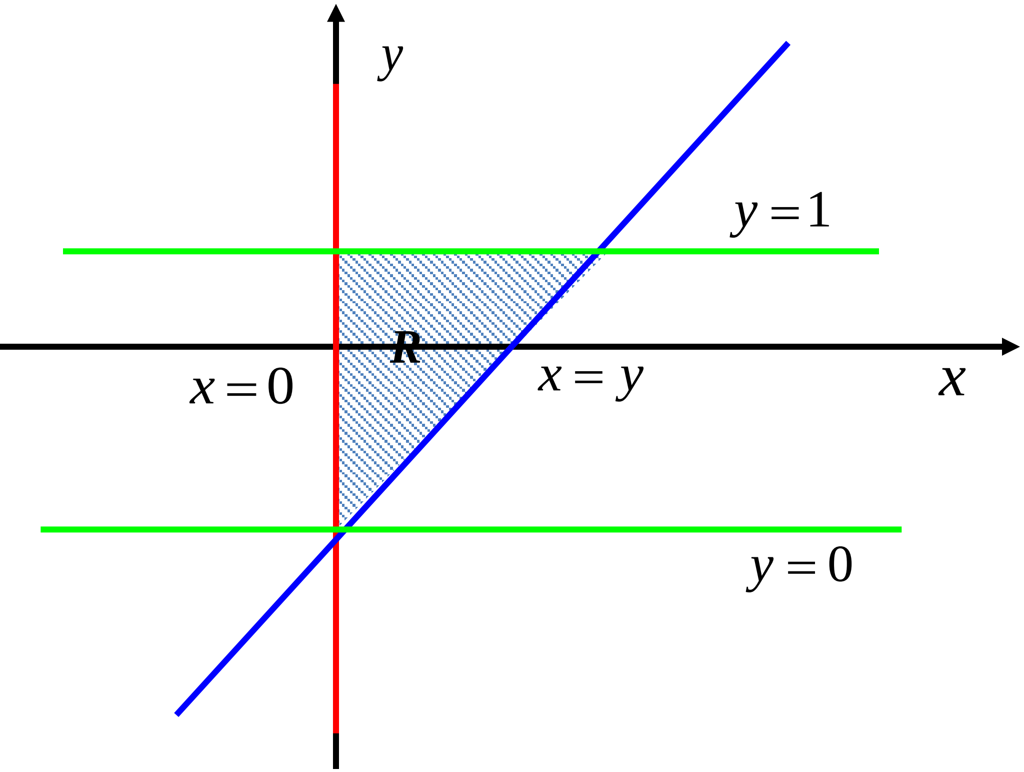
Sol:

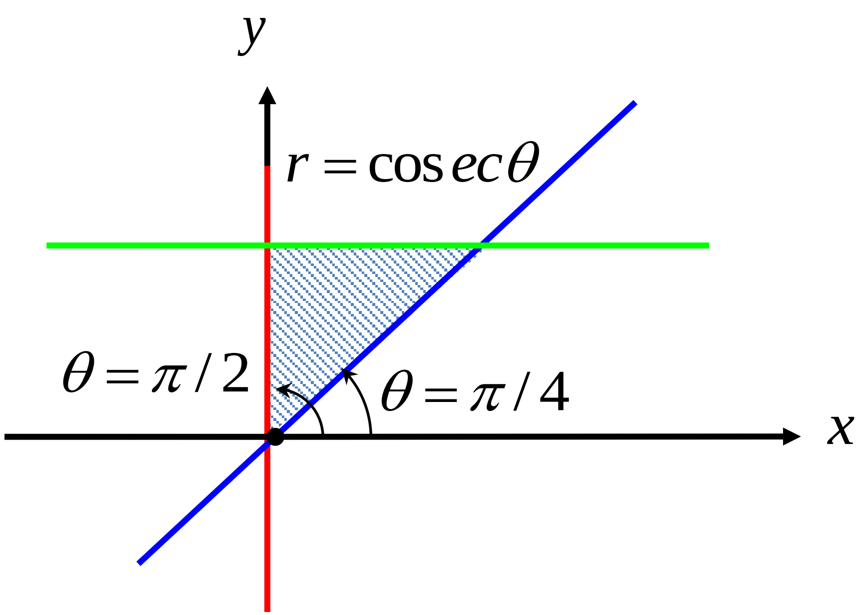
Practice: Evaluate the following integrals by changing to polar coordinates

$$\int_0^1\int_0^y \left(\frac{y^2}{x^2+y^2}\right)dx\,dy$$

Sol:

The region of integration R is described as





Therefore

$$\int_0^1 \int_0^y \left(\frac{y^2}{x^2 + y^2} \right) dx dy = \int_{\pi/4}^{\pi/2} \int_0^{\operatorname{cosec}\theta} \left(\frac{(r\sin\theta)^2}{r^2} \right) r dr d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\operatorname{cosec}\theta} r \sin^2\theta dr d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2} \right]_{r=0}^{r=\operatorname{cosec}\theta} \sin^2\theta d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2\theta}{2} \sin^2\theta d\theta$$

$$= \frac{1}{2} \int_{\pi/4}^{\pi/2} d\theta = \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{8} \quad ♣$$

Practice: Evaluate the following integrals by changing to polar coordinates

$$\int_{-3}^{3}\int_{-\sqrt{y^2-9}}^{\sqrt{y^2-9}}(4-y)\,dx\,dy$$

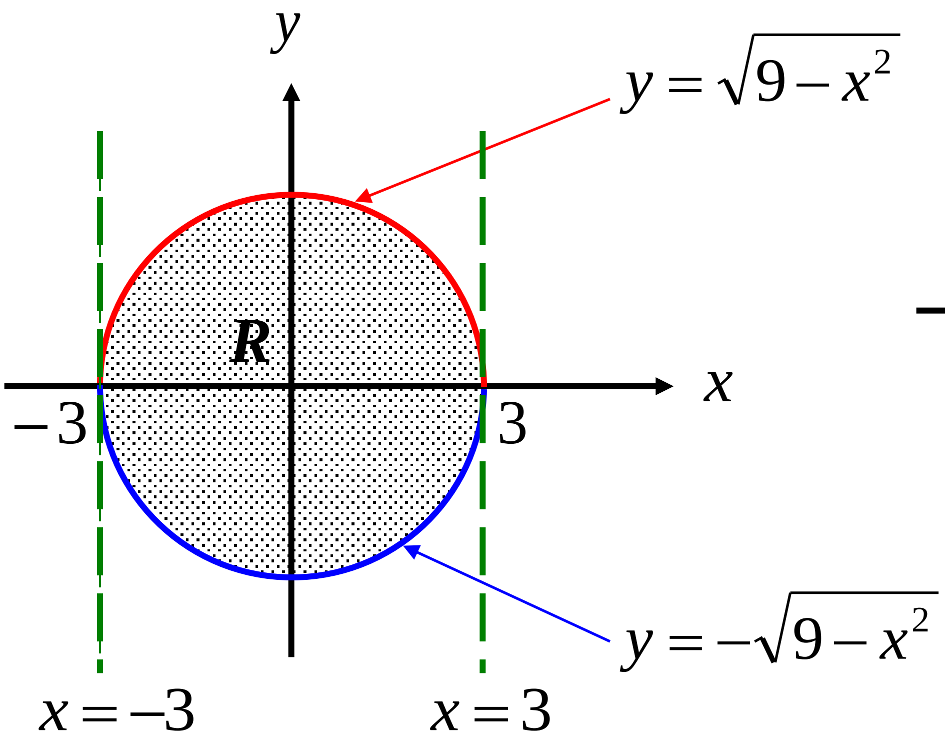
Sol:

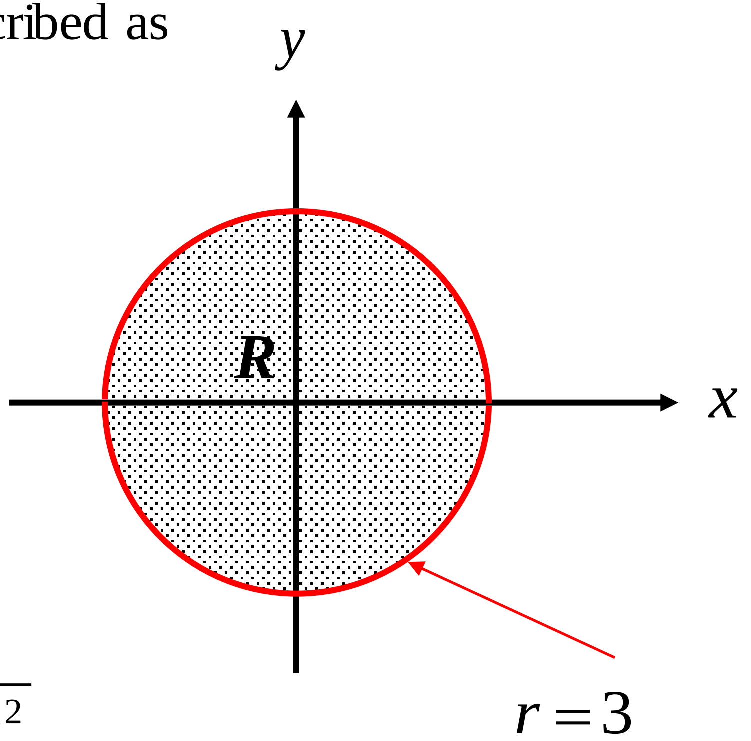
Practice: Evaluate the following integrals by changing to polar coordinates

$$\int_{-3}^{3}\int_{-\sqrt{y^2-9}}^{\sqrt{y^2-9}}(4-y)\,dx\,dy$$

Sol:

The region of integration R is described as





$$\int_{-3}^{3}\int_{-\sqrt{y^2-9}}^{\sqrt{y^2-9}}(4-y)\,dx\,dy=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3}(4-r\cos\theta)\,r\,dr\,d\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{3}(4r-r^2\cos\theta)\,dr\,d\theta$$

$$=\int_{0}^{2\pi}\left[2r^2-\frac{r^3}{3}\cos\theta\right]_{r=0}^{r=3}d\theta$$

$$=\int_{0}^{2\pi}(18-9\cos\theta)\,d\theta$$

$$=\left[18\theta-9\sin\theta\right]_{0}^{2\pi}$$

$$=36\pi$$