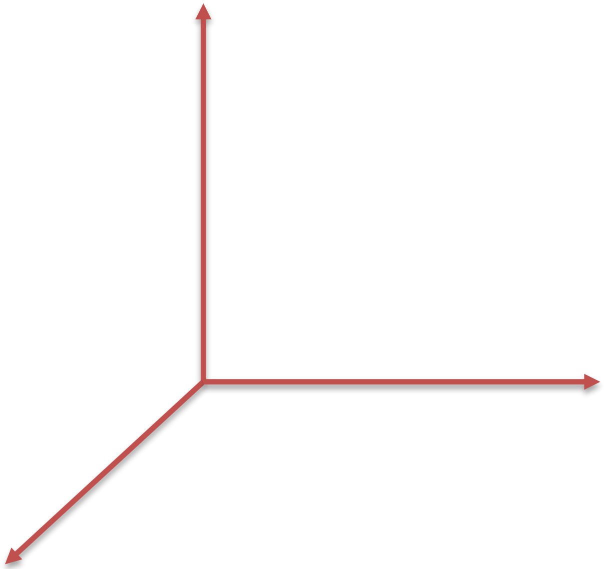
Coordinate Systems

Cartesian Coordinate System

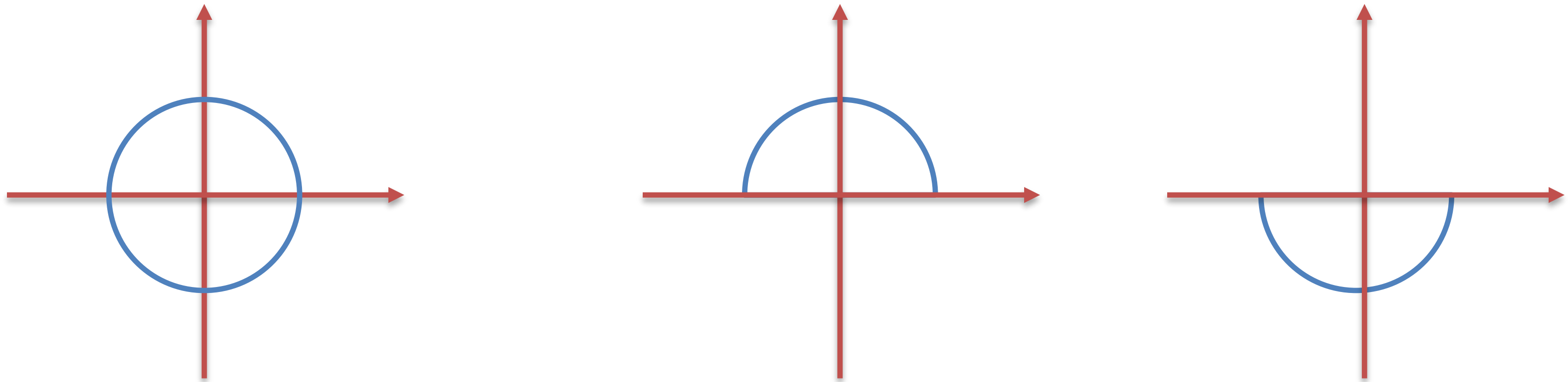
We have studied Cartesian coordinate system and sketched the graph of several multivariable functions in it.



However it is observed that certain objects are difficult to sketch in this coordinate system. The same problem even arise in 2D Cartesian plane.

Cartesian Coordinate System

For example a circle in Cartesian plane has to be cut into two parts in order to treat it as a function of one variable $y = f(x)$.



Similarly it is difficult to work with the equation $x^2 + y^2 = r^2$ for doing calculus on circle. Imagine if you were to calculate the derivative of y with respect to x and vice versa. Further the integration of function y with respect to x will not be easy either.

In order to solve this problem we identify and classify objects with respect to a given symmetry.

Symmetry

It is observed that certain objects carry an inherited ***symmetry*** which can help us to introduce a coordinate system that not only simplifies the representation of functions but also help us to visualize the function easily.

For example if someone ask you to sketch the function

$$x^2 + y^2 = \sqrt{x^2 + y^2} + x,$$

then it is not at all trivial to sketch above function. Its even harder to do calculus on this function.

This is in fact your own heart !!!

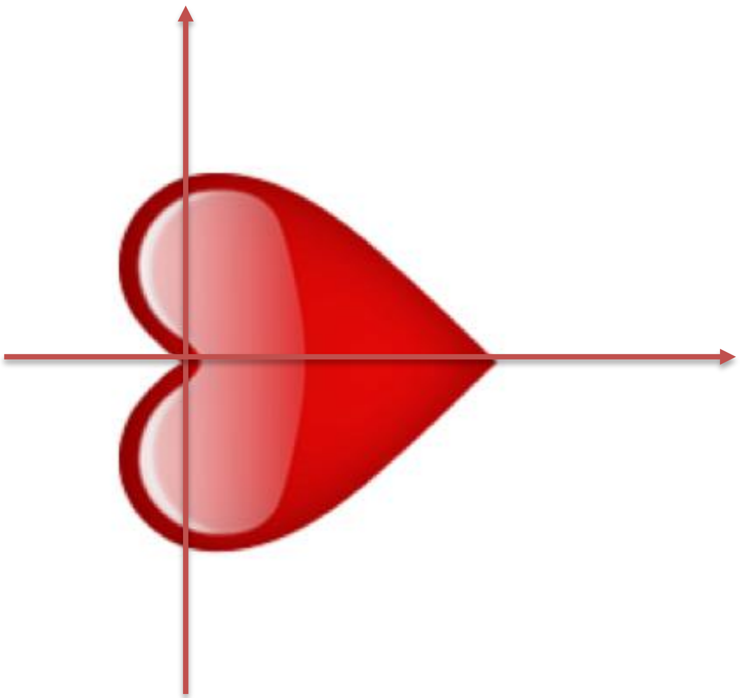
Symmetry

Q. How do we know that its heart?

Ans. By simply bringing the polar coordinate system

$$r = 1 + \cos\theta$$

heart is ***not*** as complicated in a new coordinate system as it was in the other !!!



$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 = x^2 + y^2,\ \theta = \arctan\theta$$

Symmetry

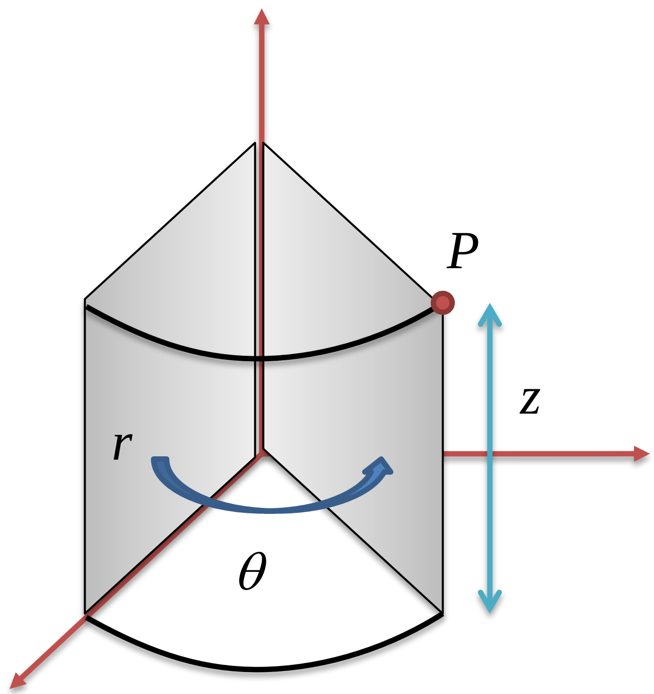
Conclusion:

In calculus, the change of coordinate systems help us to:

- calculate derivatives and integration easily
- avoid expression swelling
- understand geometry
- sketch complicated functions (especially implicit)
- comprehend underlying symmetry

Cylindrical Coordinate System

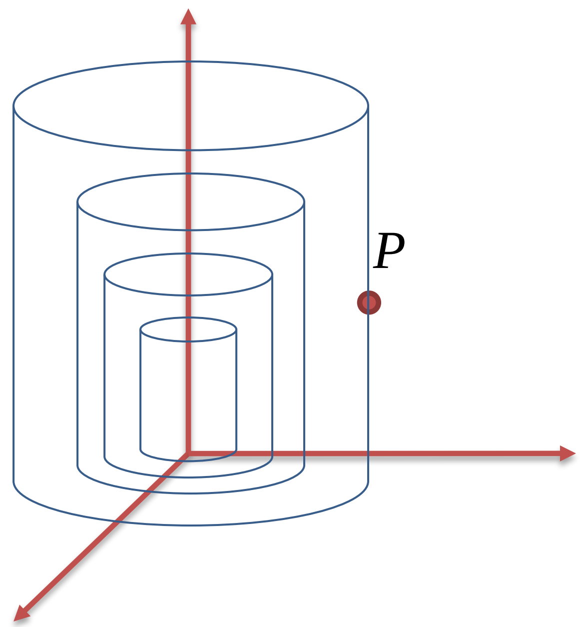
We are now in a position to naturally ***generalize*** polar coordinates in a three dimensional space. For example just make the third variable z as arbitrary to find a triplet (r, θ, z) which identify a point in space.



Now the sense of ***approaching*** a particular point is changed entirely because we reach point P via some cylinder of fixed radius and angle.

Cylindrical Coordinate System

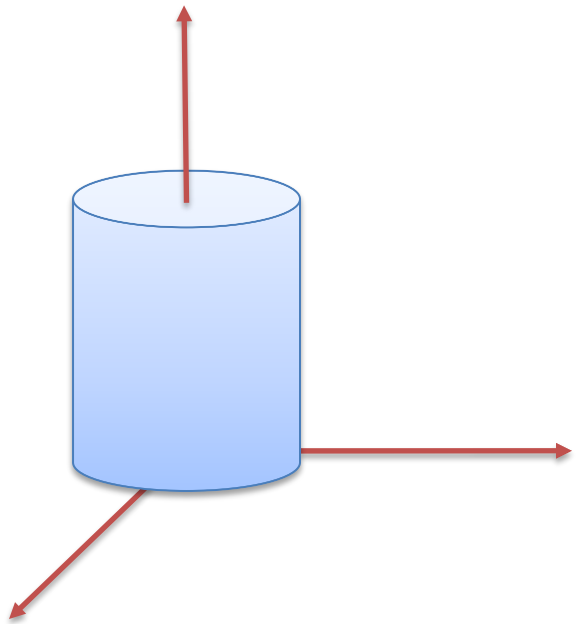
We are now in a position to naturally ***generalize*** polar coordinates in a three dimensional space. For example just make the third variable *z* as arbitrary to find a triplet (r,θ,z) which identify a point in space.



Therefore the whole three dimensional space is a bunch of infinitely many ***cylinders***.

Cylindrical Coordinate System

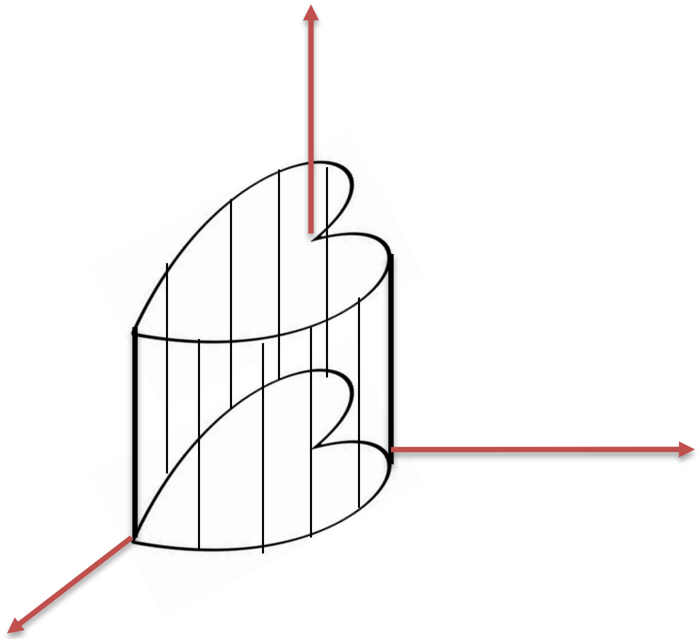
Circle in 3D: $r = 1,\ \ 0 \le z \le 1$



Notice that a circle becomes a cylinder in 3D. Therefore it would be very useful to use a cylindrical coordinate system in situations where cylindrical symmetry is present. For example, fluid flows across pipelines (mostly cylindrical), electron orbiting around nucleus, planets circular orbits around sun, the galactic structure of galaxies.

Cylindrical Coordinate System

Heart in 3D: $r = 1 + \cos\theta,\ \ 0 \le z \le 1$



Notice that it also carry a cylindrical symmetry.

Cylindrical Coordinate System

Practice Questions:

In the following questions sketch each graph and convert the given coordinate system into another.

1. $r = \sin\theta,\ \ 0 \le z \le 1$

2. $y = x,\ \ 0 \le z \le 1$

3. $y = 1,\ \ 0 \le z \le 1$

4. $r = 1,\ \ \theta = \pi / 4$

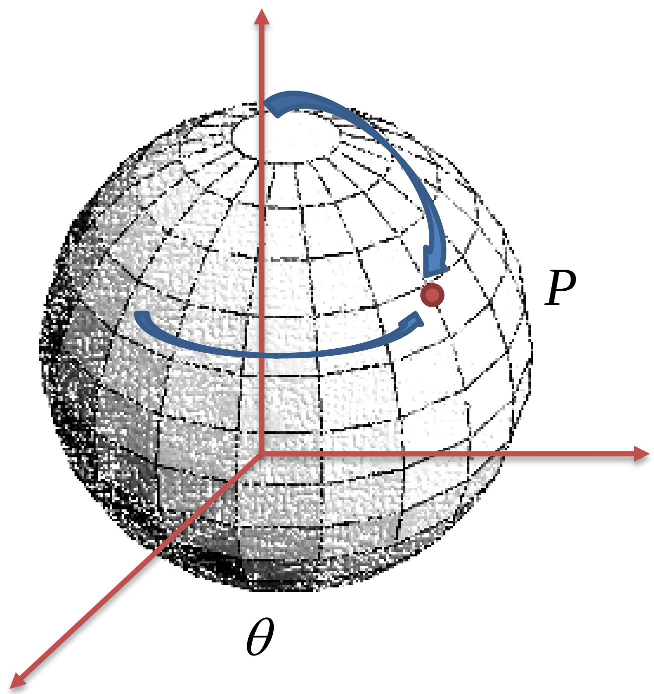
5. $r = 1,\ \ 0 \le \theta \le \pi / 4$

6. $0 \le \theta \le \pi / 4$

8. Convert a sphere of radius 1 centered at origin in cylindrical coordinates.

Cylindrical Coordinate System

We are now in a position to naturally ***generalize*** the concept of coordinate systems further in a three dimensional space by replacing the whole idea of rectangular boxes and cylinders with ***spheres*** to approach a particular point in space.



Now the sense of ***approaching*** a particular point is changed entirely because we reach point P via some sphere of fixed radius and angle.

Spherical Coordinate System

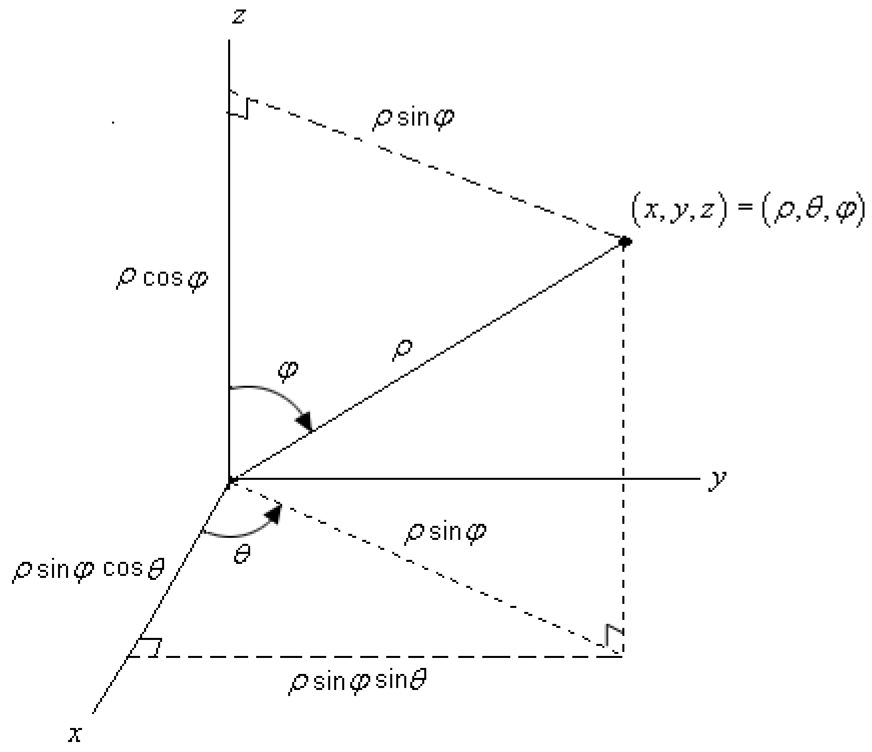
The transformation is

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2, \theta = \tan^{-1}(y/x), z = \cos^{-1}(z/\rho)$$

where

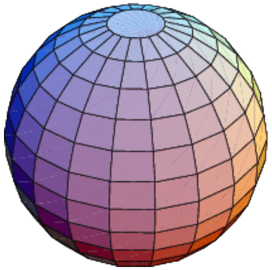
$$0 \le \rho < \infty$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi$$



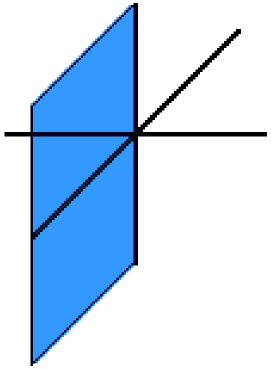
Spherical Coordinate System

Examples:

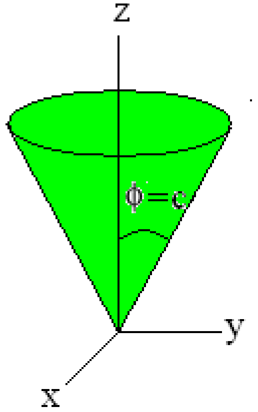
1. ρ = constant is a sphere



2. θ = constant is a plane



3. ϕ= constant is a cone



Spherical Coordinate System

Practice Questions:

In the following questions sketch each graph and convert the given coordinate system into another.

1. $z = 1$

2. $y = x, \ 0 \leq z \leq 1$

3. $y = 1, \ 0 \leq z \leq 1$

4. $\rho = 1, \ \theta = \pi / 4$

5. $\rho = 1, \ 0 \leq \theta \leq \pi / 4$

6. $0 \leq \theta \leq \pi / 4$

8. Convert a sphere of radius 1 centered at origin in spherical coordinates.