Applications

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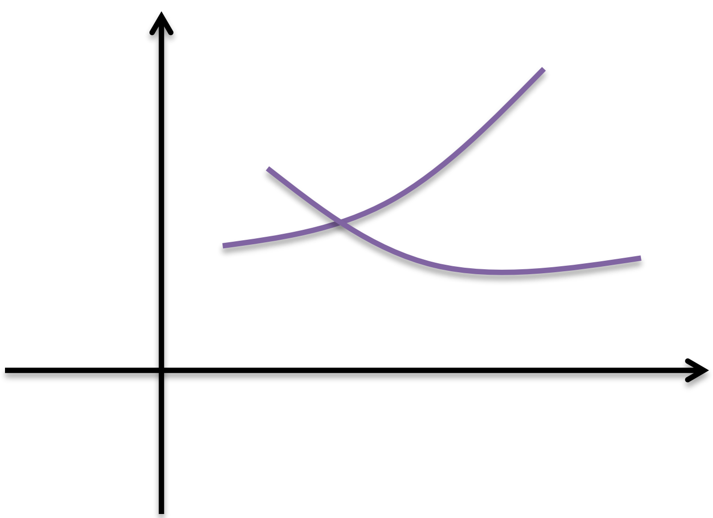
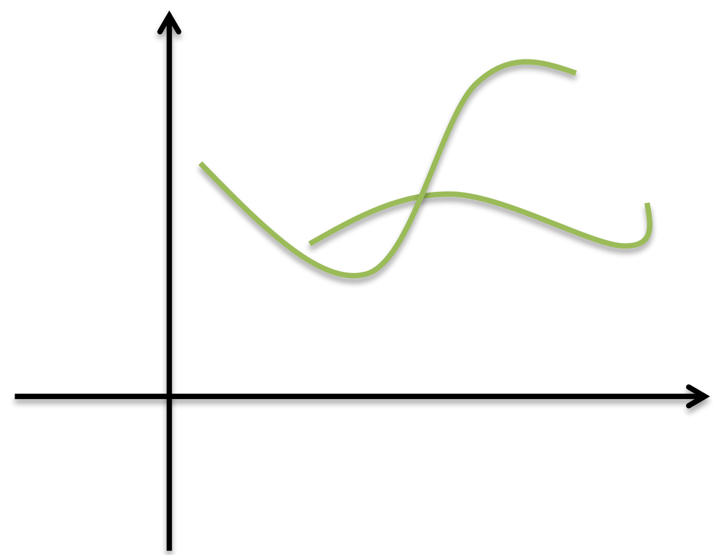


z- plane



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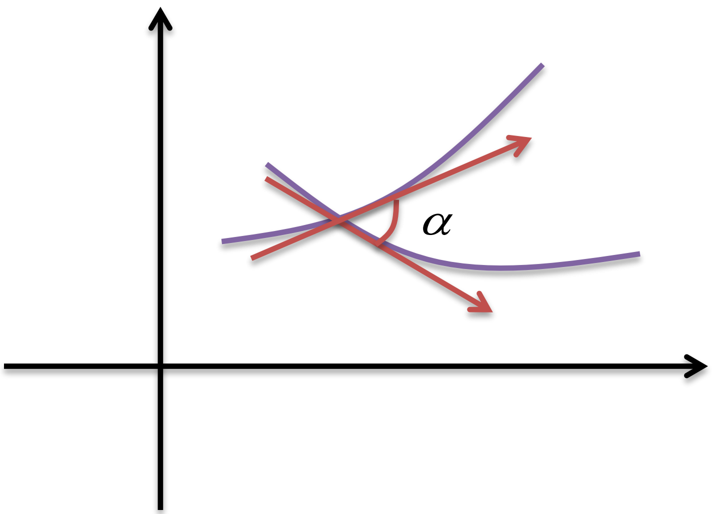

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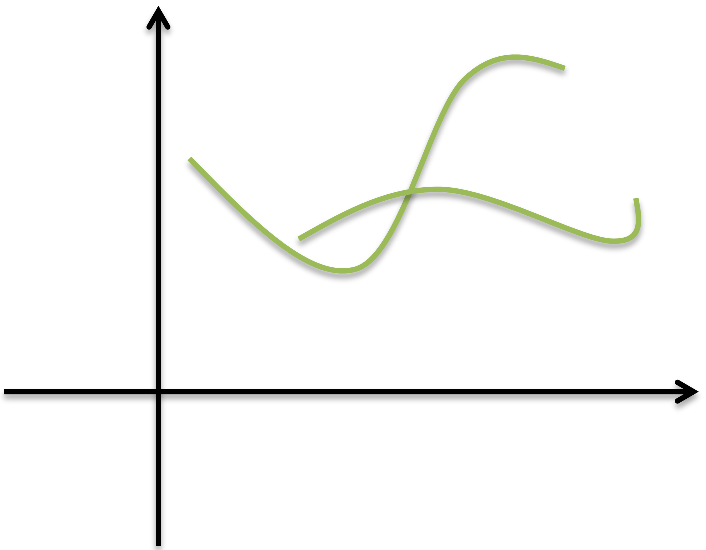

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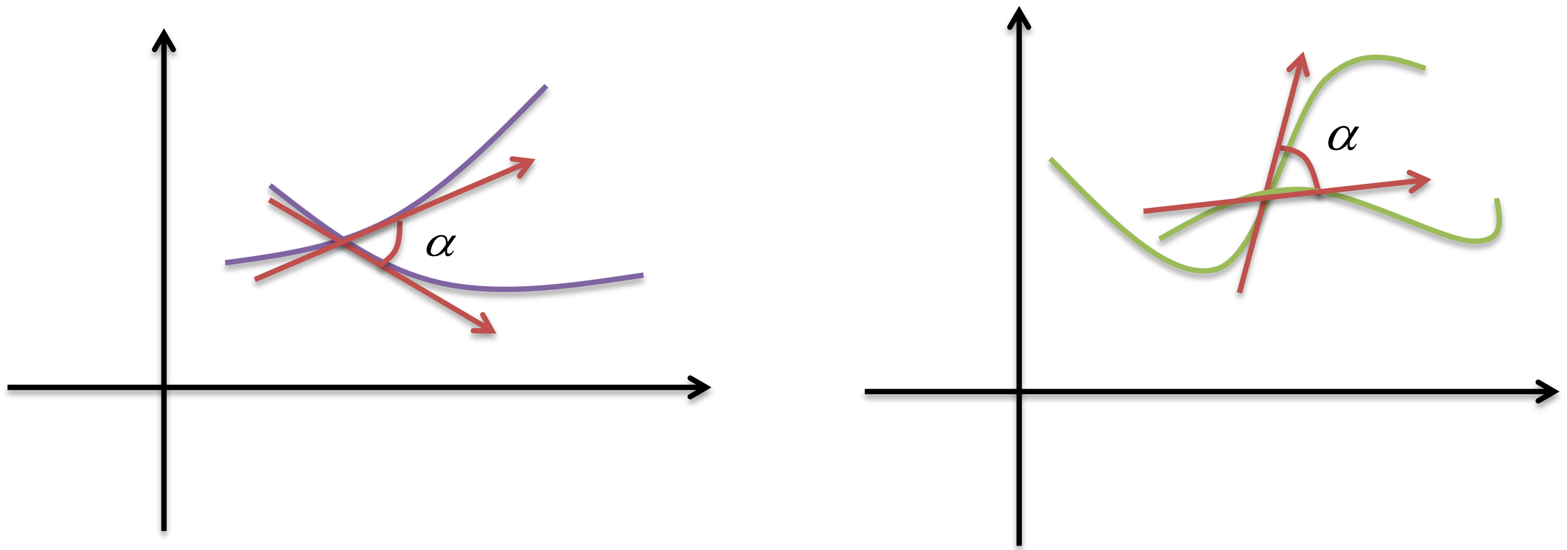

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$$\Rightarrow \quad \left(u - \tfrac{1}{2}\right)^2 + v^2 = 1$$

which is a circle in *w*- plane of radius 1 and centered at (1/2,0).

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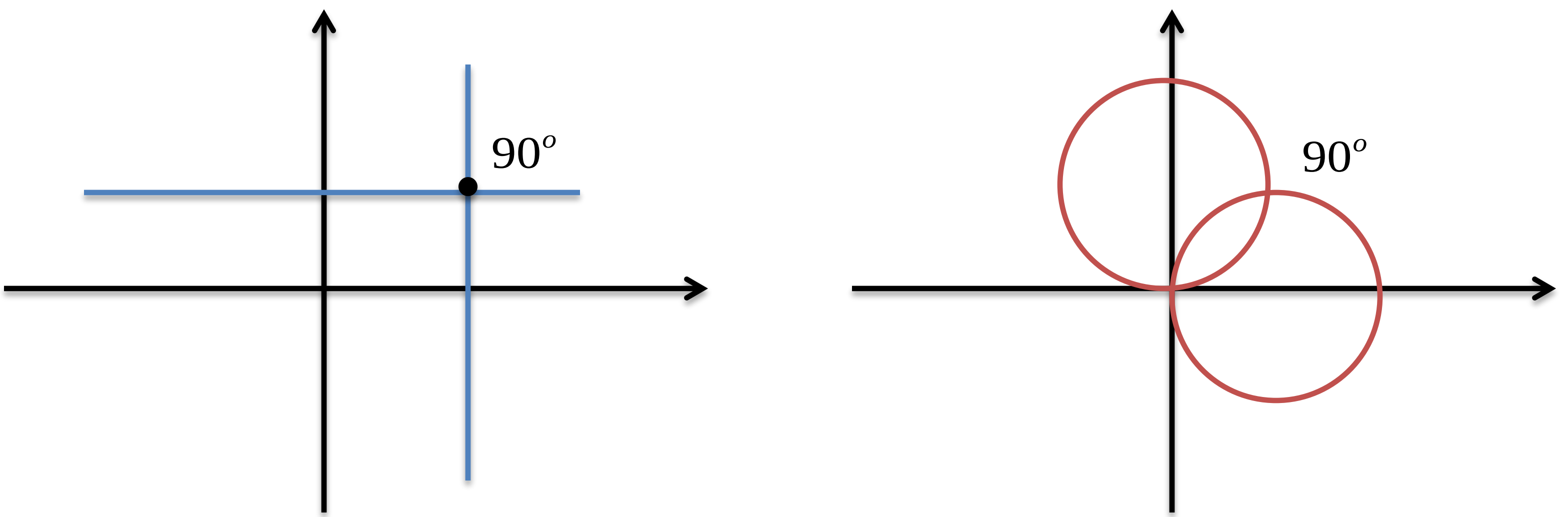

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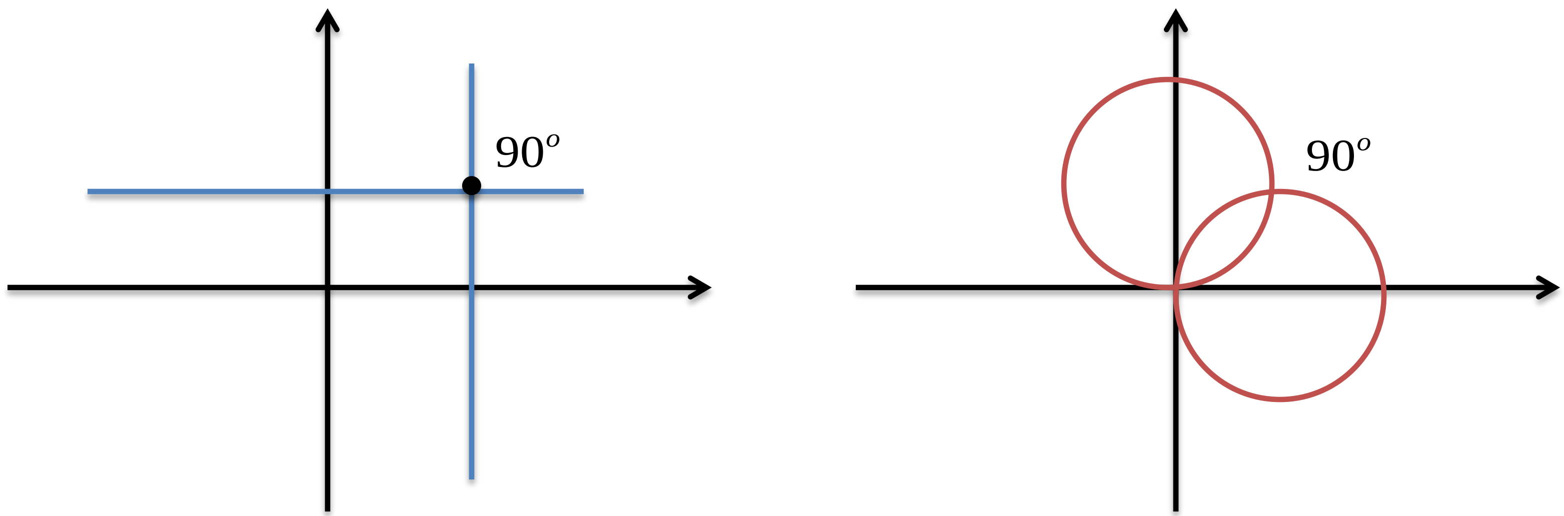

From the example we make two important observations:

- an infinite line is mapped to a closed bounded (finite) circle
- the angle 90 is preserved under complex mappings

Conformal Behavior

Therefore we conclude that

- A complex mapping can be used to map un-bounded regions to bounded regions.
- A complex analytic function respects the invariance of angle between curves.

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$$\Rightarrow \quad \varphi_{xx} + \varphi_{yy} = 0 \qquad \because \nabla \cdot \nabla = \Delta$$

Laplace Equation

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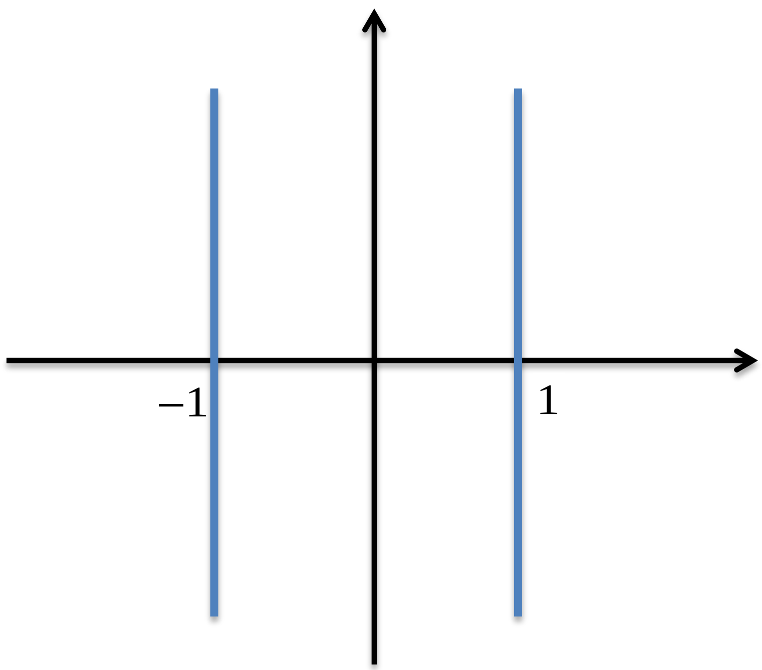

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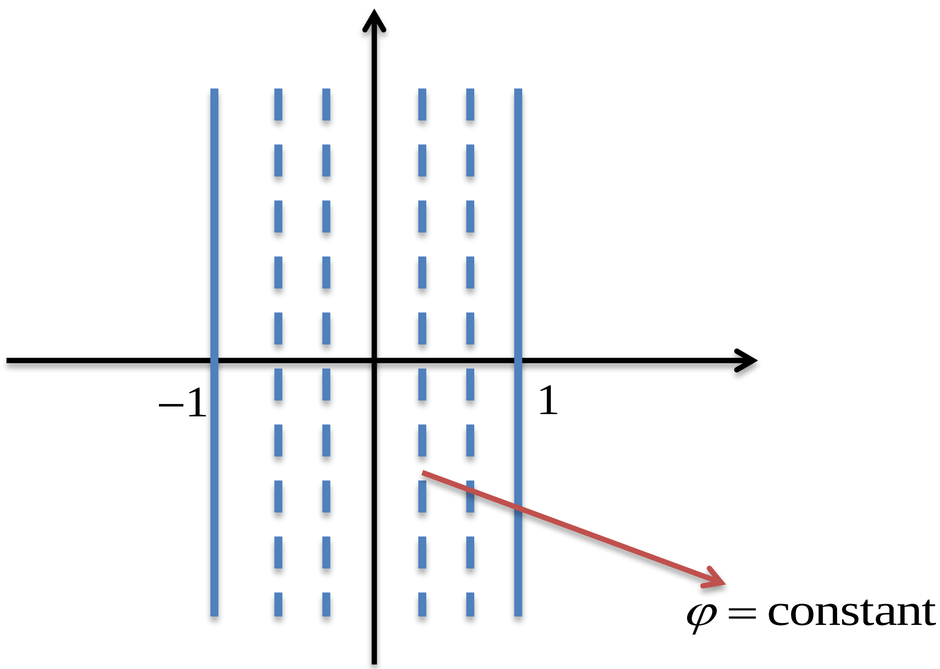

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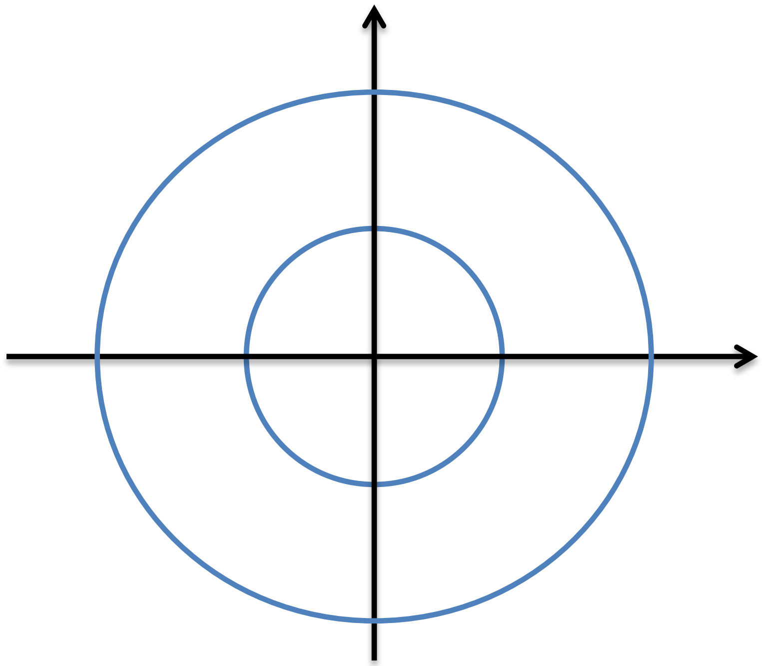

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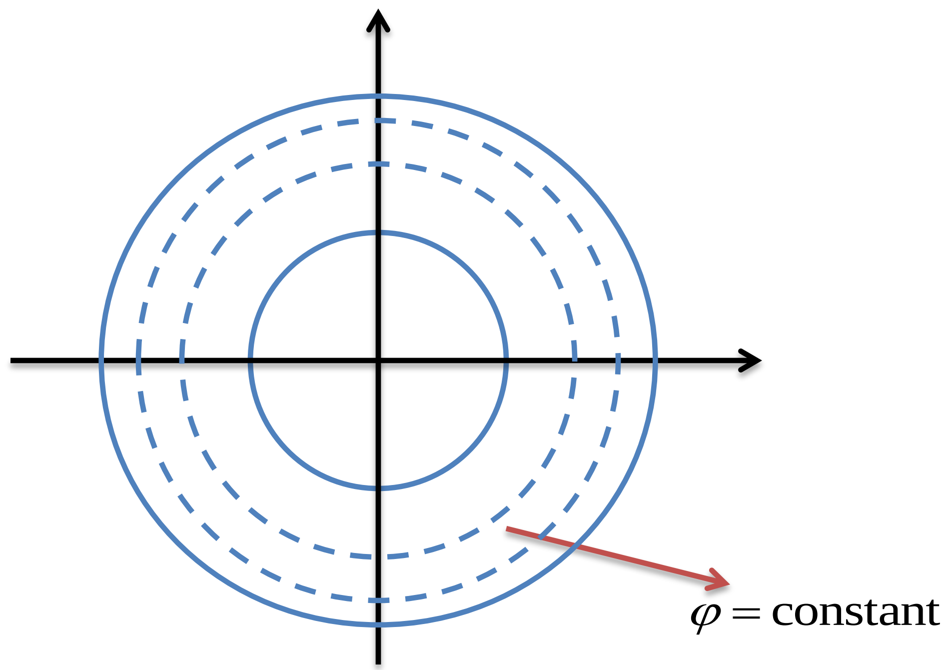

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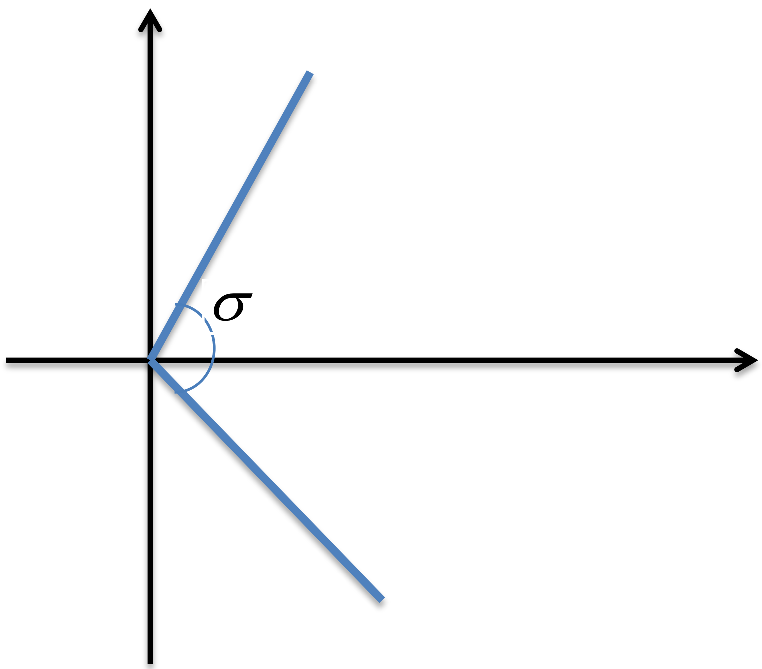

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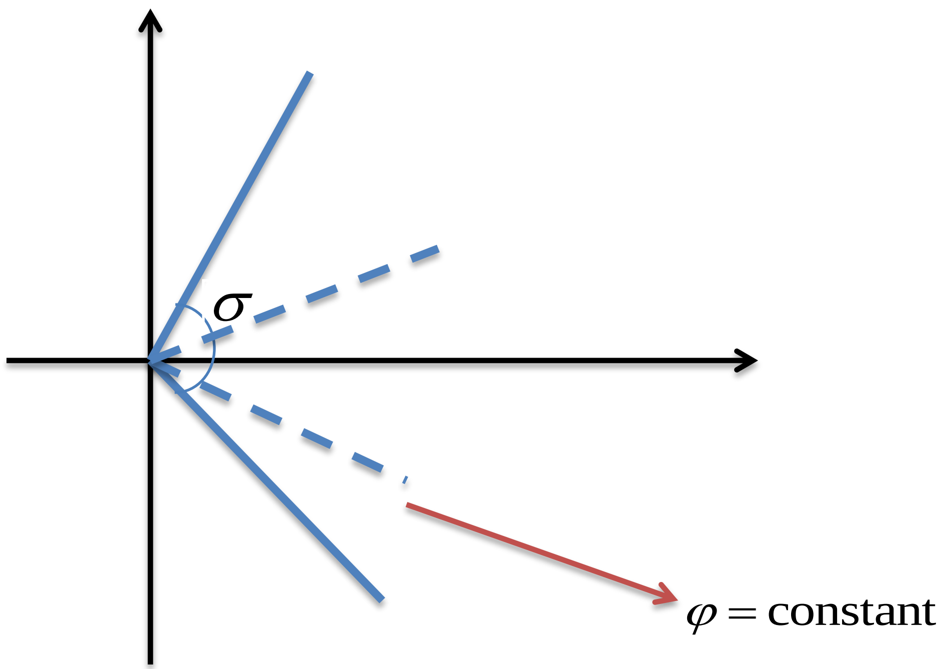

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Equipotential Surfaces **Lines of force**

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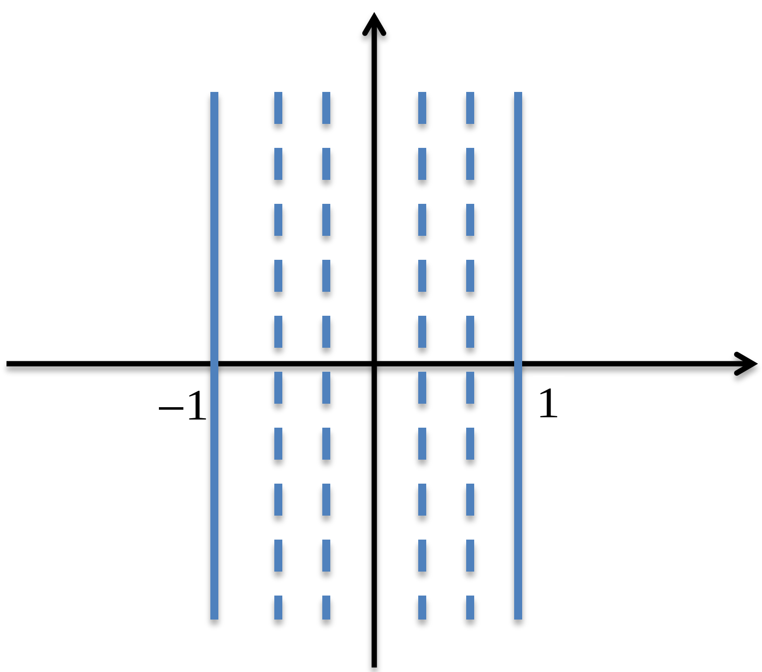

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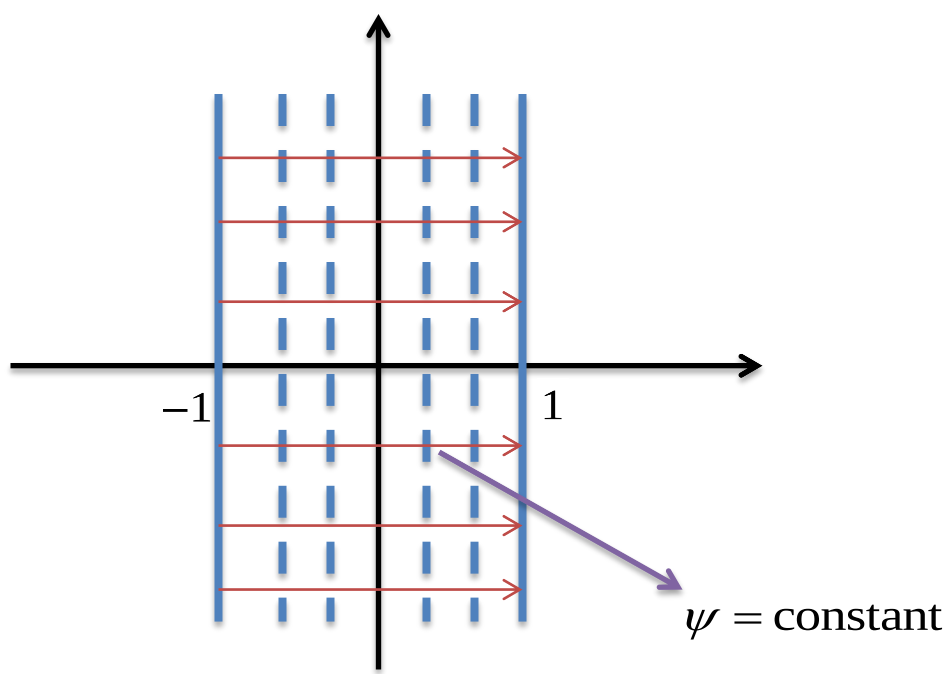

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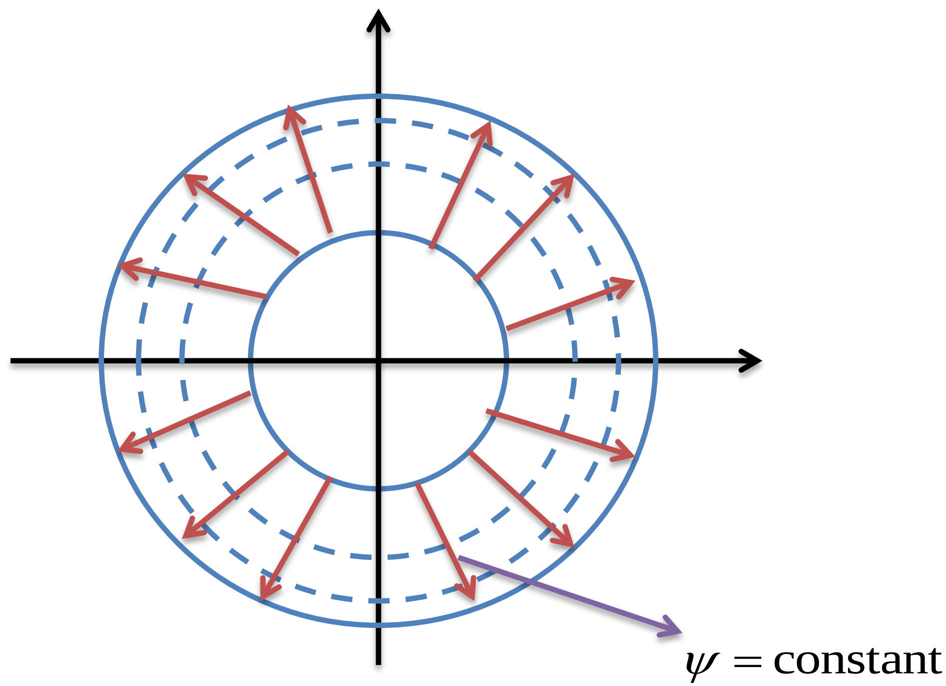

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then T = const. correspond to lines of constant temperature and are called ***isotherms*** and ψ = const. correspond to ***heat flow lines***.

Complex Potential

Example-1: Temperature field between two long parallel plates with temperatures 0 and 100 respectively, is

$$T = (100/d)\,x$$

$$\boxed{F(z) = (100/d)\,z}$$

Therefore, $\psi = (100/d)\,y = const.$