Complex Integration

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For example, complex plane, interior of a an open disk, etc.
Counter example, a unit circle without origin, an annulus region etc..

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How to evaluate complex line integral:

Way-1: Direct substitution by using fundamental theorem of complex analysis

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Way-2: Identify path and then carry out integration

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Way-1: Suppose F is analytic in a simply connected domain D then for all paths between z_0 and z_1 the fundamental theorem of complex line integrals states

$$\int_{z_0}^{z_1} F'(z)dz = F(z_1) - F(z_0)$$

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$$\frac{d}{dz}\frac{z^3}{3} = z^2$$

$$\Rightarrow \int_0^{1+i} z^2 dz = \frac{z^3}{3}\Big|_0^{1+i}$$

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Practice:

$$1. \quad \int_{-\pi i}^{\pi i} \cos z dz = ?$$

$$2. \quad \int_{-i}^{i} \frac{1}{z} dz = ?.$$

Complex Integration

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Let C be a piecewise smooth path, represented by $z = z(t)$, where $a < t < b$. Let $f(z)$ be a continuous function on C then

$$\int_c f(z)dz = \int_a^b f(z)\dot{z}(t)dt$$

where $\dot{z} = \dot{x} + i\dot{y}$.

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Step-4 Evaluate the integral

$$\int_C f(z) dz = \int_0^{2\pi} ie^{-it} e^{it} dt$$

$$= 2\pi i \qquad \text{Ans.}$$

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Follow-up Questions:

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Q-5 C is a unit circle

$$\int_C (z + z^{-1}) dz = ?$$

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2. For a closed parabolic path

$$z(t) = t + it^2, \quad 0 < t < 1$$
$$z(t) = 1 - t + i(1-t), \quad 0 < t < 1$$

evaluate the integral

$$\int_C z^2\, dz = ?$$

Complex Integral Theorem

We have observed that the answer to a complex line integral along a closed curve is zero or $2\pi i$ that is whether singularity is outside or inside the closed curve. It is the consequence of a famous theorem in complex analysis.

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If $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D

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$$(i)\ \int_C e^z dz = ?,\ (ii)\ \int_C \cos(z)dz = ?,\ (iii)\ \int_C z^n dz = ?,$$

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$$\int_C f(z)dz = 0$$

Practice: For an arbitrary closed curve C

$$\int_C \sec(z)dz = 0, \quad \int_C \frac{1}{z^2+4}dz = 0, \quad \text{C} = \text{Unit Circle}$$

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Independence of path:
If $f(z)$ is analytic in a simply connected domain D, then the integral of f is independent of path in D.

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Independence of path:
If $f(z)$ is analytic in a simply connected domain D, then the integral of f is independent of path in D.
This is an important result from the physical point of view.
If $f(z)$ has the sense of a vector field on the plane then above theorem implies that the vector field is conservative.

Complex Integral Formula

If $f(z)$ is analytic in a simply connected domain D, then

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

where C is taken counterclockwise and contains z_0.

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Example-1:

Suppose C : any contour containing $z_0 = 2$

$$\int_c \frac{e^z}{z - 2} dz =$$

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$$\begin{aligned}\int_c \frac{e^z}{z-2} dz &= 2\pi i |e^z|_{z=2} \\ &= 2\pi i e^z|_{z=2}, \quad \text{(as } e^z \text{ is analytic)} \\ &= 2\pi i e^2\end{aligned}$$

Complex Integral Formula

Practice

Suppose C : is the contours (i) $|z| = 1$ (ii) $|z-1| = 1$ (iii) $|z+1| = 1$. Evaluate

$$\int_c \frac{1}{z^2-4} dz = ?$$

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$$I = \int_C \frac{\frac{z^2+1}{z+1}}{z-1} dz$$

in which case $\therefore f(z) = \frac{z^2+1}{z+1}$ is analytic on the given contour, C, as $z = -1$ is not on C so, $f(1) = \frac{1+1}{1+1} = 1$

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$$\int_C = 2\pi i$$

Complex Integral Formula

Example-3:

Suppose we have $C: \quad |z+1| = 1$ then find the value of

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in which case $\therefore f(z) = \frac{z^2+1}{z-1}$ is analytic on the given contour, C, as $z = 1$ is not on C so, $f(-1) = \frac{2}{-2} = -1$ therefore

$$\int_C = -2\pi i$$