Complex Functions

Sajid Ali

SEECS-NUST

November 7, 2014

Contents

Contents

Contents

Contents

Contents

Contents

Complex Functions

A complex function f is a mapping that assigns a complex value $f(z)$ to each complex number z in its domain,

$$z \longrightarrow w = f(z)$$

where both z and $f(z)$ are in $\mathcal{C}$.

Complex Functions

A complex function f is a mapping that assigns a complex value $f(z)$ to each complex number z in its domain,

$$z \longrightarrow w = f(z)$$

where both z and $f(z)$ are in $\mathcal{C}$.

Examples:

1. The function $f(z) = z^2$, maps following points on the complex plane as follows

$$\begin{array}{lcl} 0 & \longrightarrow & 0 \\ 2 & \longrightarrow & 4 \\ \text{i} & \longrightarrow & \text{-1} \\ \text{-i} & \longrightarrow & \text{-1} \\ 1+\text{i} & \longrightarrow & 0+2\text{i} \\ \sqrt{i} & \longrightarrow & \text{i} \end{array}$$

Complex Functions

2. Similarly the function $f(z) = 1/z$, map points in the following manner

$$\begin{array}{lcl} 0 & \longrightarrow & 0 \\ 2 & \longrightarrow & \frac{1}{2} \\ \text{i} & \longrightarrow & \text{-i} \\ \text{-i} & \longrightarrow & \text{i} \\ 1+\text{i} & \longrightarrow & \frac{1}{2} - \frac{i}{2} \end{array}$$

Therefore in a complex mapping both input and output are collection of points on the complex plane.

Complex Functions

In general, the function $f(z)$ is complex therefore it can be decomposed into real and imaginary parts

$$w = f(z) = u(x, y) + iv(x, y)$$

where both u and v are real multi-variable functions of x, y.

Complex Functions

In general, the function $f(z)$ is complex therefore it can be decomposed into real and imaginary parts

$$w = f(z) = u(x, y) + iv(x, y)$$

where both u and v are real multi-variable functions of x, y.

Therefore a complex function just like complex numbers encode the information of two real multi-variable functions.

Complex Functions

In general, the function $f(z)$ is complex therefore it can be decomposed into real and imaginary parts

$$w = f(z) = u(x,y) + iv(x,y)$$

where both u and v are real multi-variable functions of x, y.

Therefore a complex function just like complex numbers encode the information of two real multi-variable functions.

Examples: Let us calculate u and v for the following complex functions

1. $f(z) = z^2$
2. $f(z) = \frac{1}{z}$
3. $f(z) = e^z$
4. $f(z) = sin(z)$

Examples

1. The two multi-variable functions corresponding to $w = f(z) = z^2$, can be obtained by putting on the complex glasses

$$\begin{aligned} f(z) &= z^2 \\ &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

Examples

1. The two multi-variable functions corresponding to $w = f(z) = z^2$, can be obtained by putting on the complex glasses

$$\begin{aligned} f(z) &= z^2 \\ &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

Therefore, the two real multi-variable functions are

$$u(x, y) = x^2 - y^2,$$
$$v(x, y) = 2xy.$$

Examples

1. The two multi-variable functions corresponding to $w = f(z) = z^2$, can be obtained by putting on the complex glasses

$$\begin{aligned} f(z) &= z^2 \\ &= (x+iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

Therefore, the two real multi-variable functions are

$$\begin{aligned} u(x,y) &= x^2 - y^2, \\ v(x,y) &= 2xy. \end{aligned}$$

Q. What about mapping complex points to complex points?

Examples

1. The two multi-variable functions corresponding to $w = f(z) = z^2$, can be obtained by putting on the complex glasses

$$\begin{aligned} f(z) &= z^2 \\ &= (x+iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

Therefore, the two real multi-variable functions are

$$\begin{aligned} u(x,y) &= x^2 - y^2, \\ v(x,y) &= 2xy. \end{aligned}$$

Q. What about mapping complex points to complex points? To answer that we re-visit the example of mapping of complex points under $f(z) = z^2$ using above real functions.

Examples

As discussed the function $f(z) = z^2$, maps the following points on the complex plane:

$$\begin{array}{lcl} 0 & \longrightarrow & 0 \\ 2 & \longrightarrow & 4 \\ \text{i} & \longrightarrow & \text{-1} \\ \text{-i} & \longrightarrow & \text{-1} \\ \frac{1}{i} & \longrightarrow & \text{-1} \\ 1+\text{i} & \longrightarrow & 0+2\text{i} \\ \sqrt{i} & \longrightarrow & \text{i} \end{array}$$

which can be viewed from the multi-variables functions as follows.

Examples

1. $w = f(z) = z^2 = x^2 - y^2 + 2ixy$

z		(x, y)		$u = x^2 - y^2$	$v = 2xy$	$w = z^2$
0	$\to$	$(0, 0)$	$\to$	$u = 0$	$v = 0$	$w = 0$
2	$\to$	$(2, 0)$	$\to$	$u = 4$	$v = 0$	$w = 4$
i	$\to$	$(0, 1)$	$\to$	$u = -1$	$v = 0$	$w = -1$
$-i$	$\to$	$(0, -1)$	$\to$	$u = -1$	$v = 0$	$w = -1$
$\frac{1}{i}$	$\to$	$(0, -1)$	$\to$	$u = -1$	$v = 0$	$w = -1$
$1 + i$	$\to$	$(1, 1)$	$\to$	$u = 0$	$v = 1$	$w = i$
$\sqrt{i}$	$\to$	$(?, ?)$	$\to$	$u = ?$	$v = ?$	$w = ?$

Domain & Range

The domain of a complex function $w = f(z)$, is a collection of points on the complex plane that can be used as input while range is the output produced by the function.

Domain & Range

The domain of a complex function $w = f(z)$, is a collection of points on the complex plane that can be used as input while range is the output produced by the function.

For example the domain of $f(z) = z^2$, is the entire complex plane and out put is ???.

Domain & Range

The domain of a complex function $w = f(z)$, is a collection of points on the complex plane that can be used as input while range is the output produced by the function.

For example the domain of $f(z) = z^2$, is the entire complex plane and out put is ???.

1. $f(z) = z^2, \qquad \mathcal{D}(f) = \mathcal{C}, \quad \mathcal{R}(f) = \mathcal{C}$
2. $f(z) = \frac{1}{z}, \qquad \mathcal{D}(f) = \mathcal{C} - 0, \quad \mathcal{R}(f) = ?$
3. $f(z) = e^z, \qquad \mathcal{D}(f) = ?, \quad \mathcal{R}(f) = ?$
4. $f(z) = sin(z), \qquad \mathcal{D}(f) = ?, \quad \mathcal{R}(f) = ?$

Domain & Range

The next important task is to sketch out a complex function.

Domain & Range

The next important task is to sketch out a complex function. There is an ambiguity as both input $z = x + iy$ and output $w = u + iv$ have two parts therefore it is not possible for us to sketch them on a single complex plane. Similarly three dimensions $\mathcal{R}^3$ is also not enough (Explain why !!!).

Domain & Range

The next important task is to sketch out a complex function. There is an ambiguity as both input $z = x + iy$ and output $w = u + iv$ have two parts therefore it is not possible for us to sketch them on a single complex plane. Similarly three dimensions $\mathcal{R}^3$ is also not enough (Explain why !!!).

Then what do we do now?

Ideas !!!

Domain & Range

The next important task is to sketch out a complex function. There is an ambiguity as both input $z = x + iy$ and output $w = u + iv$ have two parts therefore it is not possible for us to sketch them on a single complex plane. Similarly three dimensions $\mathcal{R}^3$ is also not enough (Explain why !!!).

Then what do we do now?

Ideas !!!

Riemann (1851) discovered a way to make a geometrical sense of $w = f(z)$, as was known for real functions.

Geometrical Understanding

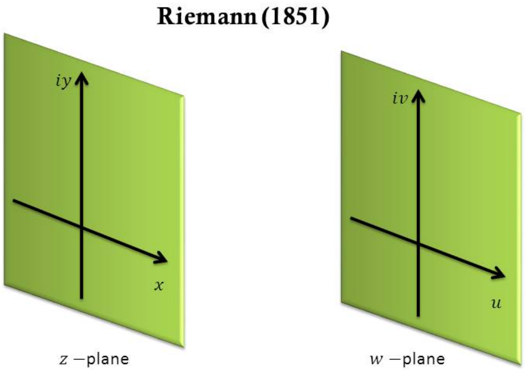
Bernhard Riemann (1826-1866) was a German mathematician who wrote his PhD thesis on complex functions. There is hardly any branch of mathematics and physics where Riemann's contributions have not made a mark.

Geometrical Understanding

Bernhard Riemann (1851) discovered a way to make a geometrical sense of $w = f(z)$. The idea is quite simple. Since both input and output of a complex function are complex numbers. Therefore a complex function may be seen as the dependence of one complex plane on another complex plane. Consequently the domain should be drawn on one complex plane ($z-$plane) and range on another complex plane ($w-$plane).

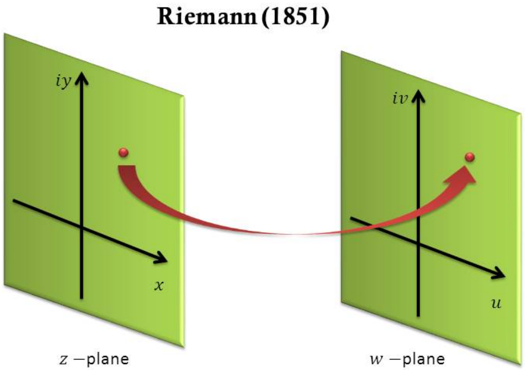
Geometrical Understanding

$w = f(z)$: The dependence of $z-$plane (input) on $w-$plane (output).



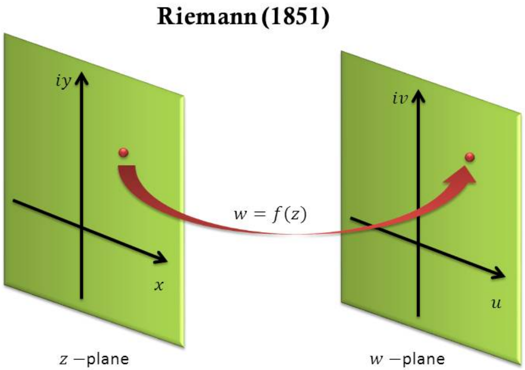
Geometrical Understanding

$w = f(z)$: The domain points in $z-$plane (input) are mapped to range points on $w-$plane (output).



Geometrical Understanding

$w = f(z)$: The domain points in $z-$plane (input) are mapped to range points on $w-$plane (output).



Examples

1. Draw the sketch of

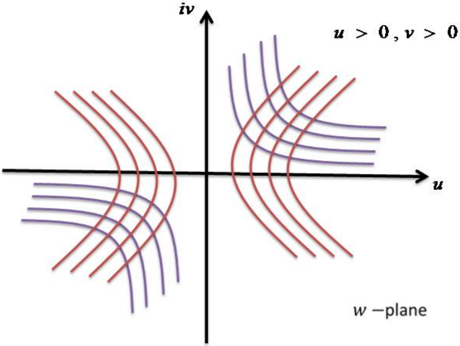
$$w = f(z) = z^2 = x^2 - y^2 + 2ixy.$$

Examples

1. Draw the sketch of $w = f(z) = z^2 = x^2 - y^2 + 2ixy$.
Draw level curves of $u, v = 0, \pm 1, \pm 2, \pm 3, \ldots$ and we see an elegant pattern of curves

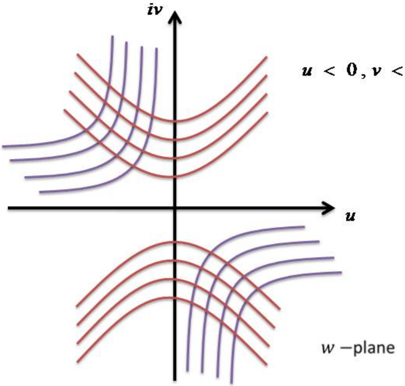
Examples

1. Draw the sketch of $w = f(z) = z^2 = x^2 - y^2 + 2ixy$.
Draw level curves of $u, v = 0, \pm 1, \pm 2, \pm 3, \ldots$ and we see an elegant pattern of curves



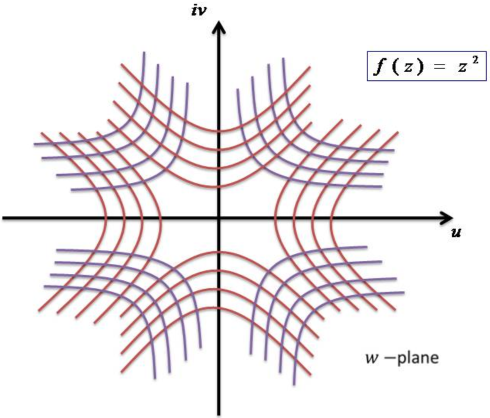
Examples

1. Draw the sketch of $w = f(z) = z^2 = x^2 - y^2 + 2ixy$.
Draw level curves of $u, v = 0, \pm 1, \pm 2, \pm 3, \ldots$ and we see an elegant pattern of curves



Examples

1. Draw the sketch of $w = f(z) = z^2 = x^2 - y^2 + 2ixy$.
Draw level curves of $u, v = 0, \pm1, \pm2, \pm3, \ldots$ and we see an elegant pattern of curves



Are they real?

Since *iota* 'i' is a familiar object for us therefore we can easily make sense of complex functions. The most important aspect of complex functions is to blend two real multi-variable functions $u(x, y)$ and $v(x, y)$.

Are they real?

Since *iota* 'i' is a familiar object for us therefore we can easily make sense of complex functions. The most important aspect of complex functions is to blend two real multi-variable functions $u(x, y)$ and $v(x, y)$. From multi-variable calculus we know that such functions do correspond to scalar fields. Therefore it born into mind which scalar fields can be the right candidates of a complex functions.

Are they real?

Since *iota* 'i' is a familiar object for us therefore we can easily make sense of complex functions. The most important aspect of complex functions is to blend two real multi-variable functions $u(x,y)$ and $v(x,y)$. From multi-variable calculus we know that such functions do correspond to scalar fields. Therefore it born into mind which scalar fields can be the right candidates of a complex functions.

Can we say that pressure and temperature of a rectangular plate can be put together in the form of a complex function?

$$f(z) = P(x,y) + iT(x,y).$$

Does it make sense?

Are they real?

Does it make sense?
No, it does not make sense because *iota* correspond to 90^o rotation therefore $iT(x, y)$ must be perpendicular to $P(x, y)$ which is not true (at least not sensible). So in
what physical situation can we think about complex functions?

Are they real?

Does it make sense?
No, it does not make sense because *iota* correspond to 90^o rotation therefore $iT(x, y)$ must be perpendicular to $P(x, y)$ which is not true (at least not sensible). So in what physical situation can we think about complex functions?

In order to develop the concepts of limit, continuity and differentiability in the complex variables it is important to see the *vicinity or neighbourhood* of a complex point.

Circles, Open & Closed disks, Domains

1. A *unit circle* in the complex domain is given by the set of points satisfying

$$|z| = 1$$

which in parametric form determined by the equation $z = e^{it}$.

Circles, Open & Closed disks, Domains

1. A *unit circle* in the complex domain is given by the set of points satisfying

$$|z| = 1$$

which in parametric form determined by the equation $z = e^{it}$.

2. An *open disk* centered at z_0 of radius ρ is a collection of interior points

$$|z - z_0| < \rho.$$

Circles, Open & Closed disks, Domains

1. A *unit circle* in the complex domain is given by the set of points satisfying

$$|z| = 1$$

which in parametric form determined by the equation $z = e^{it}$.

2. An *open disk* centered at z_0 of radius ρ is a collection of interior points

$$|z - z_0| < \rho.$$

3. A *closed disk* also contains the boundary points, i.e., $|z - z_0| \leq \rho$.

Circles, Open & Closed disks, Domains

1. A *unit circle* in the complex domain is given by the set of points satisfying

$$|z| = 1$$

which in parametric form determined by the equation $z = e^{it}$.

2. An *open disk* centered at z_0 of radius ρ is a collection of interior points

$$|z - z_0| < \rho.$$

3. A *closed disk* also contains the boundary points, i.e., $|z - z_0| \leq \rho$.

Write down the parametric form of boundary of above disk.

Limits, Continuity & Complex Derivatives

1. The limit '$l \in \mathcal{C}$' of $f(z)$ as z approaches z_0 on the complex plane is a complex number such that

$$\lim_{z \to z_0} f(z) = l.$$

Limits, Continuity & Complex Derivatives

1. The limit '$l \in \mathcal{C}$' of $f(z)$ as z approaches z_0 on the complex plane is a complex number such that

$$\lim_{z \to z_0} f(z) = l.$$

2. A complex function $f(z)$ is said to be continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Limits, Continuity & Complex Derivatives

1. The limit '$l \in \mathcal{C}$' of $f(z)$ as z approaches z_0 on the complex plane is a complex number such that

$$\lim_{z \to z_0} f(z) = l.$$

2. A complex function $f(z)$ is said to be continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

3. The complex derivative of a complex function $f(z)$ at a point z_0 in its domain is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Limits, Continuity & Complex Derivatives

1. The limit '$l \in \mathcal{C}$' of $f(z)$ as z approaches z_0 on the complex plane is a complex number such that

$$\lim_{z \to z_0} f(z) = l.$$

2. A complex function $f(z)$ is said to be continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

3. The complex derivative of a complex function $f(z)$ at a point z_0 in its domain is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Warning !!! Take care of the concept of limit as there are *infinitely many possibilities* to approach a point z_0.

Examples

1. Find the complex derivative of the function

$$f(z) = (z^3 + i)^2.$$

Examples

1. Find the complex derivative of the function

$$f(z) = (z^3 + i)^2.$$

Ans. Since it is a complex polynomial function and is differentiable therefore

$$\frac{df}{dz} = 2(z^3 + i)(3z^2)$$

Examples

2. Find the complex derivative of the function

$$f(z) = \frac{i}{(1-z)^2}, \quad z \neq 1$$

Examples

2. Find the complex derivative of the function

$$f(z) = \frac{i}{(1-z)^2}, \quad z \neq 1$$

Ans. The function has a singularity at $z = 1$, where it is not differentiable however at all other points on the punctured plane $\mathcal{C} - \{1\}$ the derivative is

$$\frac{df}{dz} = \frac{-2i}{(1-z)^3}$$

Counter Example

Check if the complex derivative of the following function exist

$$f(z) = \bar{z}$$

Ans.

Counter Example

Check if the complex derivative of the following function exist

$$f(z) = \bar{z}$$

Ans.

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

Counter Example

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} \\
&= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}
\end{aligned}
$$

Counter Example

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Follow a path along $x-$axis, where $\Delta y = 0$ then

$$f'(z) = 1$$

Counter Example

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Follow a path along $x-$axis, where $\Delta y = 0$ then

$$f'(z) = 1$$

Follow a path along $y-$axis, where $\Delta x = 0$ then

$$f'(z) = -1$$

Counter Example

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Follow a path along $x-$axis, where $\Delta y = 0$ then

$$f'(z) = 1$$

Follow a path along $y-$axis, where $\Delta x = 0$ then

$$f'(z) = -1$$

Contradiction !!! Limit if it exist has to be unique.

Counter Example

In the last example, which of the following statement is true

1. the derivative fails to exist at a particular point in the domain?

Counter Example

In the last example, which of the following statement is true

1. the derivative fails to exist at a particular point in the domain?
2. the derivative exist at a few points in the domain?

Counter Example

In the last example, which of the following statement is true

1. the derivative fails to exist at a particular point in the domain?
2. the derivative exist at a few points in the domain?
3. the derivative does not exist anywhere on the entire complex plane?

Counter Example

In the last example, which of the following statement is true

1. the derivative fails to exist at a particular point in the domain?
2. the derivative exist at a few points in the domain?
3. the derivative does not exist anywhere on the entire complex plane?

Comments:

1. Which physical operation correspond to $f(z) = \bar{z}$?

Counter Example

In the last example, which of the following statement is true

1. the derivative fails to exist at a particular point in the domain?
2. the derivative exist at a few points in the domain?
3. the derivative does not exist anywhere on the entire complex plane?

Comments:

1. Which physical operation correspond to $f(z) = \bar{z}$?
2. What about functions of $\bar{z}$?