Complex Algebra

Sajid Ali

SEECS-NUST

September 7, 2015

Contents

Complex Numbers

A combination of two real numbers coupled with *iota* 'i' forms a *complex* number

$$z = x + iy$$

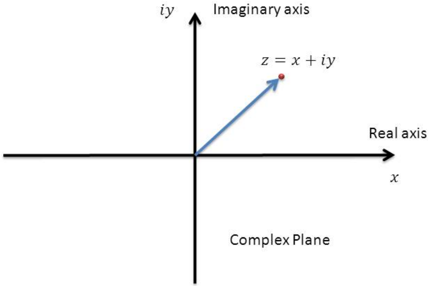
where $x \in \Re, y \in \Re$.

Complex Numbers

A combination of two real numbers coupled with *iota* 'i' forms a *complex* number

$$z = x + iy$$

where $x \in \Re, y \in \Re$.



Complex Numbers

A combination of two real numbers coupled with *iota* 'i' forms a *complex* number

$$z = x + iy$$

where $x \in \Re, y \in \Re$. Since both x and y are two real numbers so they correspond to two real quantities.

Complex Numbers

A combination of two real numbers coupled with *iota* 'i' forms a *complex* number

$$z = x + iy$$

where $x \in \Re, y \in \Re$. Since both x and y are two real numbers so they correspond to two real quantities. Hence a complex number encodes the information of two real quantities which gives them an edge over real numbers.

Complex Numbers

A combination of two real numbers coupled with *iota* 'i' forms a *complex* number

$$z = x + iy$$

where $x \in \Re, y \in \Re$. Since both x and y are two real numbers so they correspond to two real quantities. Hence a complex number encodes the information of two real quantities which gives them an edge over real numbers. This concept will be more transparent as we proceed but before that we first see some of the important properties of complex numbers.

1. It is clear that while working with complex numbers we avoid to deal with two real operations in separate.

Polar Form

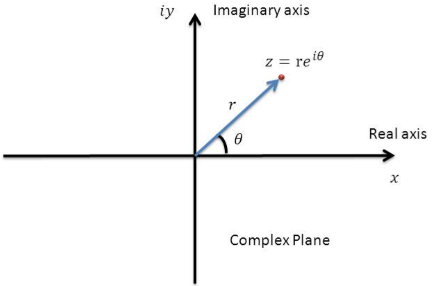
2. The importance of complex numbers can also be seen by writing them in polar form

$$z = re^{i\theta}$$

Polar Form

2. The importance of complex numbers can also be seen by writing them in polar form

$$z = re^{i\theta}$$



Polar Form

2. The importance of complex numbers can also be seen by writing them in polar form

$$z = re^{i\theta}$$

where r is known as the absolute value of z and θ is the polar angle defined by

$$r^2 = x^2 + y^2, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Polar Form

2. The importance of complex numbers can also be seen by writing them in polar form

$$z = re^{i\theta}$$

where r is known as the absolute value of z and θ is the polar angle defined by

$$r^2 = x^2 + y^2, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Notice that

$$\text{Principal argument} = \text{Arg}(z) = \theta \in (-\pi, \pi]$$
$$\text{argument} = \text{arg(z)} = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$

Brief Example

Q. Find the polar form of the complex number $z = -1 - i$.

Brief Example

Q. Find the polar form of the complex number $z = -1 - i$.
Sol. The absolute value r is

$$r = \sqrt{1+1} = \sqrt{2}\,.$$

Brief Example

Q. Find the polar form of the complex number $z = -1 - i$.
Sol. The absolute value r is

$$r = \sqrt{1+1} = \sqrt{2}.$$

In order to find the principal argument we first calculate the angle 'α', in the first quadrant

$$\alpha = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Brief Example

Q. Find the polar form of the complex number $z = -1 - i$.

Sol. The absolute value r is

$$r = \sqrt{1+1} = \sqrt{2}.$$

In order to find the principal argument we first calculate the angle 'α', in the first quadrant

$$\alpha = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Now we find out the quadrant in which the complex number lies. Since, both real and imaginary parts of z, are negative therefore it is the third quadrant. In the third quadrant we assume

$$Arg(z) = \pi + \alpha = 5\pi/4, \qquad (or - 3\pi/4)$$

Therefore, the required polar form is

$$z = \sqrt{2}\, e^{i\,5\pi/4}.$$

Practice

Q.2 Find the polar form of the complex numbers

$$i) \qquad z = \frac{1}{1-i},$$
$$ii) \qquad z = \frac{i}{(1-i)},$$
$$iii) \qquad z = (1-i)^2.$$

I am iota 'i'

I am iota 'i'

Do you remember how to realize $\sqrt{2}$, an irrational number with a non-repetitive decimal sequence that can not be seen in reality but we learnt how to see it with naked eye. Today you will see that iota 'i' will introduce itself. All we need to do is to re-tune our brains to see it in reality.

I am iota 'i'

Do you remember how to realize $\sqrt{2}$, an irrational number with a non-repetitive decimal sequence that can not be seen in reality but we learnt how to see it with naked eye. Today you will see that iota 'i' will introduce itself. All we need to do is to re-tune our brains to see it in reality.

For example every complex number can be transformed into a polar form so is $z = i$, we obtain

$$i = e^{i\pi/2}$$

therefore in reality i correspond to rotation by 90^o degrees which is why the imaginary part is always sketched on the $y-$axis as it is at 90^o to $x-$axis.

Complex Magic

Suppose we have two complex numbers in polar forms $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then their product is

$$z_1 z_2 = \underbrace{r_1 r_2}_{\text{re-scaling}} \times \underbrace{e^{i(\theta_1+\theta_2)}}_{\text{rotation}}$$

Complex Magic

Suppose we have two complex numbers in polar forms $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then their product is

$$z_1 z_2 = \underbrace{r_1 r_2}_{\text{re-scaling}} \times \underbrace{e^{i(\theta_1+\theta_2)}}_{\text{rotation}}$$

Therefore we conclude that

$$\text{complex product} = \underbrace{\text{re-scaling} + \text{rotation}}_{\text{two real operations together}}$$

Complex Magic

Suppose we have two complex numbers in polar forms $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then their product is

$$z_1 z_2 = \underbrace{r_1 r_2}_{\text{re-scaling}} \quad \times \quad \underbrace{e^{i(\theta_1+\theta_2)}}_{\text{rotation}}$$

Therefore we conclude that

$$\text{complex product} = \quad \underbrace{\text{re-scaling} + \text{rotation}}_{\text{two real operations together}}$$

a complex operation encodes an information of two real physical operations re-scaling and rotation. Now think on similar lines about other complex operations e.g., addition, division, complex conjugation.

Complex Magic

Let us explore the complex domain in more detail. Consider a real quadratic equation

$$x^2 - 1 = 0.$$

Complex Magic

Let us explore the complex domain in more detail. Consider a real quadratic equation

$$x^2 - 1 = 0.$$

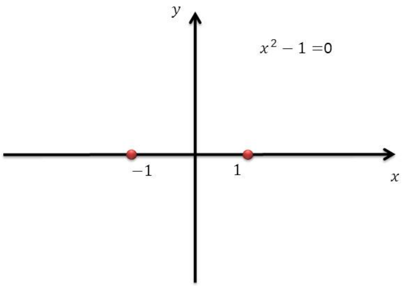
We know that the solutions of this equation are $x = \pm 1$, which are two points on the real line.

Complex Magic

Let us explore the complex domain in more detail. Consider a real quadratic equation

$$x^2 - 1 = 0.$$

We know that the solutions of this equation are $x = \pm 1$, which are two points on the real line.



Complex Magic

Now consider the complex equation

$$z^2 - 1 = 0.$$

Complex Magic

Now consider the complex equation

$$z^2 - 1 = 0.$$

How many solution does this complex equation has?

Complex Magic

Now consider the complex equation

$$z^2 - 1 = 0.$$

How many solution does this complex equation has? Of course two $z = \pm 1$, two points on the complex plane.

Complex Magic

Now consider the complex equation

$$z^2 - 1 = 0.$$

How many solution does this complex equation has? Of course two $z = \pm 1$, two points on the complex plane. If the answers are same then you might believe that both equations are same !!!.

Complex Magic

Now consider the complex equation

$$z^2 - 1 = 0.$$

How many solution does this complex equation has? Of course two $z = \pm 1$, two points on the complex plane. If the answers are same then you might believe that both equations are same !!!. It is certainly *not* true.

Complex Magic

The beauty of second equation can only be seen if we put on the *complex glasses*

Complex Magic

The beauty of second equation can only be seen if we put on the *complex glasses*, which means if we substitute $z = x + iy$, into the second equation

$$(x + iy)^2 - 1 = 0$$
$$x^2 - y^2 + 2ixy - 1 = 0$$
$$x^2 - y^2 - 1 + i2xy = 0 + i0$$

Complex Magic

The beauty of second equation can only be seen if we put on the *complex glasses*, which means if we substitute $z = x + iy$, into the second equation

$$
\begin{aligned}
&(x+iy)^2 - 1 = 0 \\
&x^2 - y^2 + 2ixy - 1 = 0 \\
&x^2 - y^2 - 1 + i2xy = 0 + i0
\end{aligned}
$$

which gives two equations

$$
\begin{aligned}
&x^2 - y^2 = 1, \\
&2xy = 0, \qquad \text{Hmhmmm interesting !!!}
\end{aligned}
$$

Complex Magic

The beauty of second equation can only be seen if we put on the *complex glasses*, which means if we substitute $z = x + iy$, into the second equation

$$\begin{aligned}(x + iy)^2 - 1 &= 0\\ x^2 - y^2 + 2ixy - 1 &= 0\\ x^2 - y^2 - 1 + i2xy &= 0 + i0\end{aligned}$$

which gives two equations

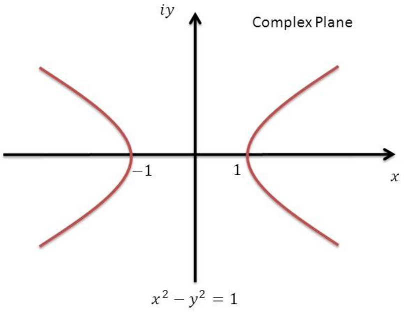
$$x^2 - y^2 = 1,$$

$$2xy = 0, \qquad \text{Hmhmmm interesting !!!}$$

These are two equations which have a beautiful geometry. The first is an equation of a *hyperbola* and the other is an equation of $x-$ *or* $y-$*axis*.

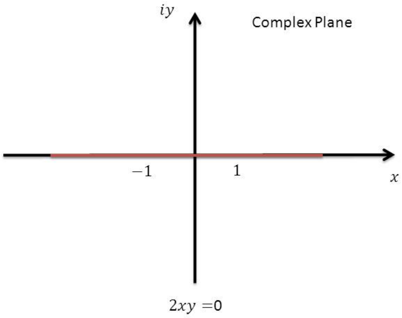
Complex Magic

These are two equations which have a beautiful geometry. The first is an equation of a *hyperbola* and the other is an equation of *$x-$ or $y-$axis*.



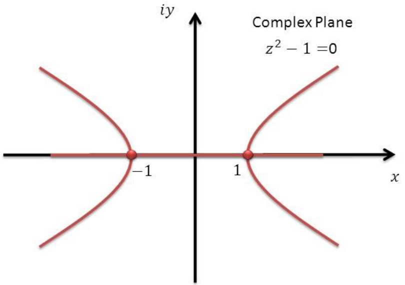
Complex Magic

These are two equations which have a beautiful geometry. The first is an equation of a *hyperbola* and the other is an equation of *$x-$ or $y-$axis*.



Complex Magic

Therefore it is only after when we wear complex glasses we could see the two solutions $z = \pm 1$, are basically two points on the *intersection* of hyperbolas and $x - axis$.



Complex Magic

Put on your complex glasses and study a linear and a quadratic complex equation

$$z + i = 1$$
$$z^2 + 1 = 0.$$

What do they represent analytically? Discuss the geometry behind these equations. Compare the result with real equations, i.e., in which case $z = x$.

Complex Algebra

We are familiar with the concept of a vector space which in simple words is an elegant pattern followed by the set of numbers or objects.

Complex Algebra

We are familiar with the concept of a vector space which in simple words is an elegant pattern followed by the set of numbers or objects. The two basic properties of a vector space are

$$v_1 + v_2 \in \mathcal{V}, \quad \forall \quad v_1, v_2 \in \mathcal{V}$$
$$\alpha v \in \mathcal{V}, \quad \forall \quad \alpha \in \mathcal{R}$$

Complex Algebra

We are familiar with the concept of a vector space which in simple words is an elegant pattern followed by the set of numbers or objects. The two basic properties of a vector space are

$$v_1 + v_2 \in \mathcal{V}, \quad \forall \quad v_1, v_2 \in \mathcal{V}$$
$$\alpha v \in \mathcal{V}, \quad \forall \quad \alpha \in \mathcal{R}$$

For example the space of all real numbers $\mathcal{R}$, is a vector space. We know that if we add or multiply two real numbers then the result is again a real number. On the other hand, $0 \in \mathcal{R}$, is a number which if we add into another real number gives the same real number. Similarly $1 \in \mathcal{R}$, is a multiplicative identity.

Complex Algebra

We are familiar with the concept of a vector space which in simple words is an elegant pattern followed by the set of numbers or objects. The two basic properties of a vector space are

$$v_1 + v_2 \in \mathcal{V}, \quad \forall \quad v_1, v_2 \in \mathcal{V}$$
$$\alpha v \in \mathcal{V}, \quad \forall \quad \alpha \in \mathcal{R}$$

For example the space of all real numbers $\mathcal{R}$, is a vector space. We know that if we add or multiply two real numbers then the result is again a real number. On the other hand, $0 \in \mathcal{R}$, is a number which if we add into another real number gives the same real number. Similarly $1 \in \mathcal{R}$, is a multiplicative identity.

Q. What about the set of complex numbers?

Complex Algebra

Interestingly the set of complex numbers ($\mathcal{C}$), follows the same pattern of a vector space, i.e.

Complex Algebra

Interestingly the set of complex numbers ($\mathcal{C}$), follows the same pattern of a vector space, i.e.

$$z_1 + z_2 \in \mathcal{C}, \quad \forall \quad z_1, z_2 \in \mathcal{C}$$
$$\alpha z \in \mathcal{C}, \quad \forall \quad \alpha \in \mathcal{R}.$$

Complex Algebra

Interestingly the set of complex numbers ($\mathcal{C}$), follows the same pattern of a vector space, i.e.

$$z_1 + z_2 \in \mathcal{C}, \quad \forall \quad z_1, z_2 \in \mathcal{C}$$
$$\alpha z \in \mathcal{C}, \quad \forall \quad \alpha \in \mathcal{R}.$$

It is theoretically important because it brings complex numbers on equal footing to the set of real numbers. However there are drastic differences between the two number systems. For example in real numbers we can say that $2 < 100$, but in complex numbers it is *illogical* to write $2 + i < 100$ or $i < 2i$. Explain why?.

Complex Algebra

Some of the important facts about complex numbers are

- $\mathcal{C}$ is a complex vector space.
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2} \qquad \bar{z} = x - iy$
- $z = re^{i\theta}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $e^{i\theta} = \cos\theta + i\sin\theta$, Euler Identity (Prove it!)

Complex Algebra

Some of the important facts about complex numbers are

- An $n-$th degree complex polynomial equation has n complex roots. On the other hand an $n-$th degree real polynomial equation may or may not have n real roots.

Formulae

- $\text{Log} z = \ln r + i\theta$
- $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$
- $\sin(i\theta) = i\sinh(\theta), \quad \cos(i\theta) = \cosh(\theta)$
- $z^{1/n} = r^{1/n} \exp\left(\frac{\theta + 2k\pi}{n} i\right), \quad k = 0, 1, 2, ..., n-1$

Practice Questions

1. 'ι' is nothing but rotation by $\pi/2$. Verify it for $z = 1 + i$.

2. If $z = x + iy$, find $\text{Re}(1/\bar{z})$.

3. Convert $z = 1 + i, 1 - i, -1 + i, -1 - i$, into polar forms.

4. Find the cube root of unity in the complex domain. Comment on roots of a complex algebraic equation.

5. $\text{Log}(i) =?$. Comment on the domain of complex Logarithm function.

Applications

The complex numbers disturbed the intuitions of scientists for quite long time. In fact the founder of calculus of complex variables *Riemann*, had a lot of trouble getting his PhD thesis approved in the mid of the nineteenth century. Today there is hardly any branch of science where his contributions have not left a mark. Below I list a few fields where his methods are rigorously used.

Engineering:
Electrical Network Analysis, Signals and Systems, Communication Systems, etc.

Mathematical Physics:
Potential Theory, Fluid Mechanics, Quantum Mechanics, String Theory, Mandle-brot sets, Fractals, etc.