Analytic Functions
&
Conformal Mappings

Contents

- Analytic Function
- Cauchy Riemann Equations
- Laplace Equation & Harmonic Conjugate
- Conformal Functions
- Physical Relevance
- Applications

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Remark: The CR-equations are not only necessary condition for a function to be analytic but are also sufficient condition.

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Note that

$$u_x = v_y , \quad u_y = -v_x$$

Therefore the function is analytic except at $z = 1$.

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Since the above relation is valid at all points in the domain therefore the function is not analytic anywhere in the z - plane.

Analytic Functions

Practice: Determine if the following complex function are analytic.

$$(i)\quad f(z) = e^{z} \qquad (ii)\quad f(z) = \sin(z)$$

$$(iii)\quad f(z) = e^{\bar{z}} \qquad (iv)\quad f(z) = |z|^2$$

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Polar Coordinates: For complicated complex functions it is better to use Euler formula which require that the CR-equations are obtained in $\left(r, \bar{\ }\right)$. The Cauchy-Riemann equations in polar coordinates are given by

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Practice: Determine if the following complex function are analytic.

$$(i) \quad f(z) = z^6 \qquad (ii) \quad f(z) = \log(z)$$

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Caution: For above practice questions we first need to transform both u and v in terms of the polar coordinates (r,θ).

Harmonic Conjugate

If $f(z) = u + iv$, is analytic in a domain D, then partial derivatives of *u* and *v* exist at each point of D and satisfy Cauchy-Riemann equations

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The function *u* is known as the harmonic conjugate of *v* and vice versa. We are now in a position to determine which multi-variable functions are candidates of complex functions. So the question arises.

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Answer: No

Question: Can we find harmonic conjugate of every multivariable function?

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Answer: No

Reason:

Suppose we are given a real multi-variable function then in order to find its harmonic conjugate we can employ CR-equations and in this way we would be able to obtain a complex function for which these two functions become real and imaginary parts. However it is not possible to obtain harmonic conjugate of every real multi-variable function because both of them must satisfy the necessary condition that they have *continuous first order* derivatives in a given domain.

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Integrating above equation with respect to x

$$u = e^{x} \cos y + \ddot{\ } (y)$$

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$$v_y = e^x \cos y$$

$$\Rightarrow \quad u_x = e^x \cos y$$

Integrating above equation with respect to x

$$\Rightarrow \quad u = e^x \cos y + \ddot{\ } (y)$$

Now differentiate it with respect to y and use the other CR equation

$$\Rightarrow \quad u_y = -e^x \cos y + \frac{d \ddot{\ } (y)}{dy}$$

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$$v(x, y) = e^x \sin y$$

Solution:

$$u_y = -e^x \cos y + \frac{d\ddot{\ }(y)}{dy}$$

$$-v_x = -e^x \cos y + \frac{d\ddot{\ }(y)}{dy}$$

$$-e^x \cos y = -e^x \cos y + \frac{d\ddot{\ }(y)}{dy}$$

$$\frac{d\ddot{\ }(y)}{dy} = 0$$

$$\ddot{\ }(y) = \text{constant}$$

Harmonic Conjugate

Example - 1: Find the harmonic conjugate of

$$\boxed{v(x, y) = e^{x} \sin y}$$

Solution:

$$\Rightarrow \quad \ddot{\ } (y) = \text{constant} = \text{c}$$

$$\Rightarrow \quad u = e^{x} \cos y + c$$

which gives us the harmonic conjugate and it can be used to find complex function

$$\Rightarrow \quad f(z) = u + i v$$

$$\Rightarrow \quad f(z) = e^{x} \cos y + i e^{x} \sin y + c$$

$$\Rightarrow \quad f(z) = e^{x} \cdot e^{i y} + c$$

$$\Rightarrow \quad f(z) = e^{z} + c$$

Harmonic Conjugate

Practice: Find the harmonic conjugate of these

$$(i) \quad u = e^{2x}\sin 2y \qquad (ii) \quad u = x^2 - y^2$$

Laplace Equation

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Differentiate both equations with respect to *x*

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Similarly we obtain Laplace equation in *v*

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Similarly we obtain Laplace equation in *v*

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Therefore an analytic complex function includes two real multi-variables functions which satisfy Laplace equation.

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Example - 1: Find the value of k for which u satisfy Laplace equation

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Solution: Differentiate twice u with respect to x and y

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$$u_{yy} = -4 e^{kx} \sin(2y)$$

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$$\Rightarrow \quad k^2 - 4 = 0$$

$$\Rightarrow \quad k = \pm 2$$

Laplace Equation

Example - 1: Find the value of k for which u satisfy Laplace equation

$$u(x,y) = e^{kx}\sin(2y)$$

Solution: Now as we know that u satisfy Laplace equation for

$$u(x,y) = e^{2x}\sin(2y)$$

Question: Can we find the harmonic conjugate of this function?

Question: What about the corresponding complex function?