Complex Algebra

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SEECS-NUST

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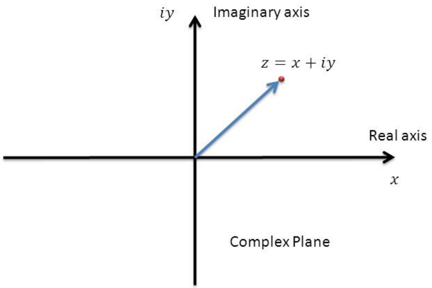

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where $x \in \Re, y \in \Re$. Since both x and y are two real numbers so they correspond to two real quantities. Hence a complex number encodes the information of two real quantities which gives them an edge over real numbers. This concept will be more transparent as we proceed but before that we first see some of the important properties of complex numbers.

1. It is clear that while working with complex numbers we avoid to deal with two real operations in separate.

Polar Form

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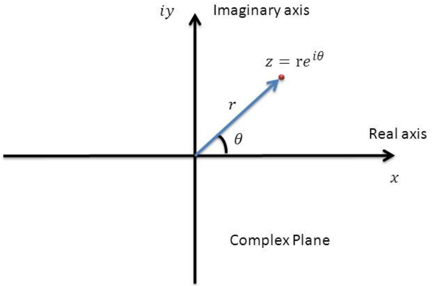

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Notice that

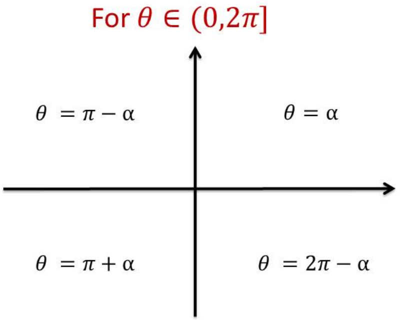
$$\text{Principal argument} = \text{Arg}(z) = \theta \in (-\pi, \pi]$$
$$\text{argument} = \text{arg(z)} = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}$$

An Easy Way To Determine Angle

For any given complex number $z = x + iy$, we ignore all of its signs and find the angle

$$\alpha = \arctan(y/x), \text{ such that } x \geq 0, y \geq 0$$

then we determine actual angle by first identifying the quadrant in which z resides and find θ using diagram



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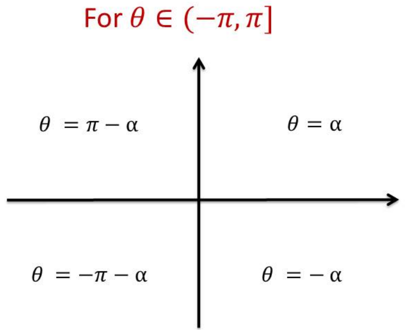

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Now we find out the quadrant in which the complex number lies. Since, both real and imaginary parts of z, are negative therefore it is the third quadrant. In the third quadrant we assume

$$Arg(z) = \pi + \alpha = 5\pi/4, \qquad (or - 3\pi/4)$$

Therefore, the required polar form is

$$z = \sqrt{2}\; e^{i\,5\pi/4}.$$

Practice

Q.2 Find the polar form of the complex numbers

$$
\begin{aligned}
i) \qquad & z = \frac{1}{1-i}, \\
ii) \qquad & z = \frac{i}{(1-i)}, \\
iii) \qquad & z = (1-i)^2.
\end{aligned}
$$

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Do you remember how to realize $\sqrt{2}$, an irrational number with a non-repetitive decimal sequence that can not be seen in reality but we learnt how to see it with naked eye. Today you will see that iota 'i' will introduce itself. All we need to do is to re-tune our brains to see it in reality.

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For example every complex number can be transformed into a polar form so is $z = i$, we obtain

$$i = e^{i\pi/2}$$

therefore in reality i correspond to rotation by 90^o degrees which is why the imaginary part is always sketched on the $y-$axis as it is at 90^o to $x-$axis.

Complex Magic

Suppose we have two complex numbers in polar forms $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then their product is

$$z_1 z_2 = \underbrace{r_1 r_2}_{\text{re-scaling}} \times \overbrace{e^{i(\theta_1+\theta_2)}}_{}$$

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a complex operation encodes an information of two real physical operations re-scaling and rotation. Now think on similar lines about other complex operations e.g., addition, division, complex conjugation.

Complex Magic

Since every complex number can be expressed in a polar form which, however, depends on the use of Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The proof of Euler formula is available everywhere. An important observation comes from the question: what is the phase difference between sine and cosine waves? Quickly, you will find out that it is $90^o = \pi/2$, which opens up a new direction to realize iota. Therefore, an iota provides a coupling between a sine and a cosine wave in a unified and unique way.

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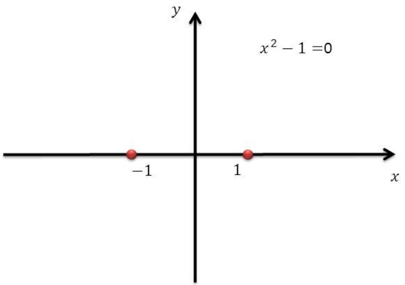

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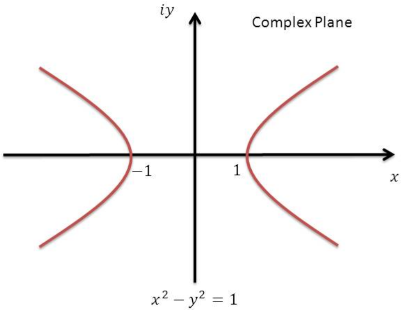

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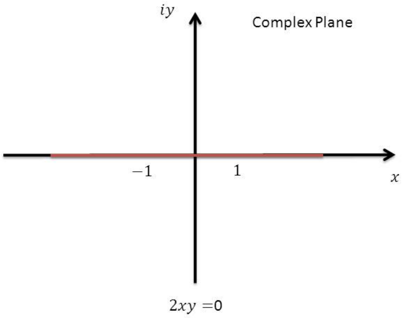
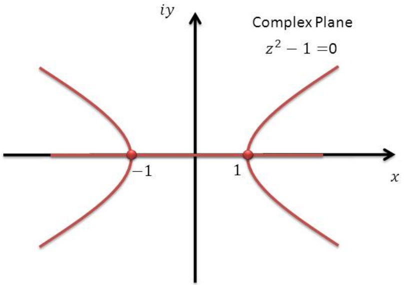


Complex Magic

Therefore it is only after when we wear complex glasses we could see the two solutions $z = \pm 1$, are basically two points on the *intersection* of hyperbolas and $x - axis$.



Practice

1. Put on your complex glasses and study a linear and a quadratic complex equation

$$z + i = 1$$
$$z^2 + 1 = 0.$$

Discuss the geometry behind these equations. Compare the result with real equations, i.e., in which case $z = x$.
2. Consider the following equation $z^3 - 1 = 0$, which has three solutions given by

$$1, \quad \frac{-1 \pm \sqrt{3}i}{2}.$$

First obtain these solutions using the nth-root formula for finding roots of unity. Then, wear your complex glasses to make real sense of above complex numbers. Identify curves which gives you these solutions in the intersection.

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For example the space of all real numbers $\mathcal{R}$, is a vector space. We know that if we add or multiply two real numbers then the result is again a real number. On the other hand, $0 \in \mathcal{R}$, is a number which if we add into another real number gives the same real number. Similarly $1 \in \mathcal{R}$, is a multiplicative identity.

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Q. What about the set of complex numbers?

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It is theoretically important because it brings complex numbers on equal footing to the set of real numbers. However there are drastic differences between the two number systems. For example in real numbers we can say that $2 < 100$, but in complex numbers it is *illogical* to write $2 + i < 100$ or $i < 2i$. Explain why?.

Complex Algebra

Some of the important facts about complex numbers are

- $\mathcal{C}$ is a complex vector space.
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2} \qquad \bar{z} = x - iy$
- $z = re^{i\theta}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $e^{i\theta} = \cos\theta + i\sin\theta,$ Euler Identity (Prove it!)

Complex Algebra

Some of the important facts about complex numbers are

- An $n-$th degree complex polynomial equation has n complex roots. On the other hand an $n-$th degree real polynomial equation may or may not have n real roots.

Formulae

- $\mathrm{Log} z = \ln r + i\theta$
- $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$
- $\sin(i\theta) = i\sinh(\theta), \quad \cos(i\theta) = \cosh(\theta)$
- $z^{1/n} = r^{1/n} \exp\left(\frac{\theta + 2k\pi}{n} i\right), \quad k = 0, 1, 2, ..., n-1$

Practice Questions

1. 'ι' is nothing but rotation by $\pi/2$. Verify it for $z = 1 + i$.

2. If $z = x + iy$, find $\text{Re}(1/\bar{z})$.

3. Convert $z = 1 + i, 1 - i, -1 + i, -1 - i$, into polar forms.

4. Find the cube root of unity in the complex domain. Comment on roots of a complex algebraic equation.

5. $\text{Log}(i) = ?$. Comment on the domain of complex Logarithm function.

Applications

The complex numbers disturbed the intuitions of scientists for quite long time. In fact the founder of calculus of complex variables *Riemann*, had a lot of trouble getting his PhD thesis approved in the mid of the nineteenth century. Today there is hardly any branch of science where his contributions have not left a mark. Below I list a few fields where his methods are rigorously used.

Engineering:
Electrical Network Analysis, Signals and Systems, Communication Systems, etc.

Mathematical Physics:
Potential Theory, Fluid Mechanics, Quantum Mechanics, String Theory, Mandle-brot sets, Fractals, etc.