Novel Construction Methods of
Quaternion Orthogonal Designs
Based on Complex Orthogonal Designs

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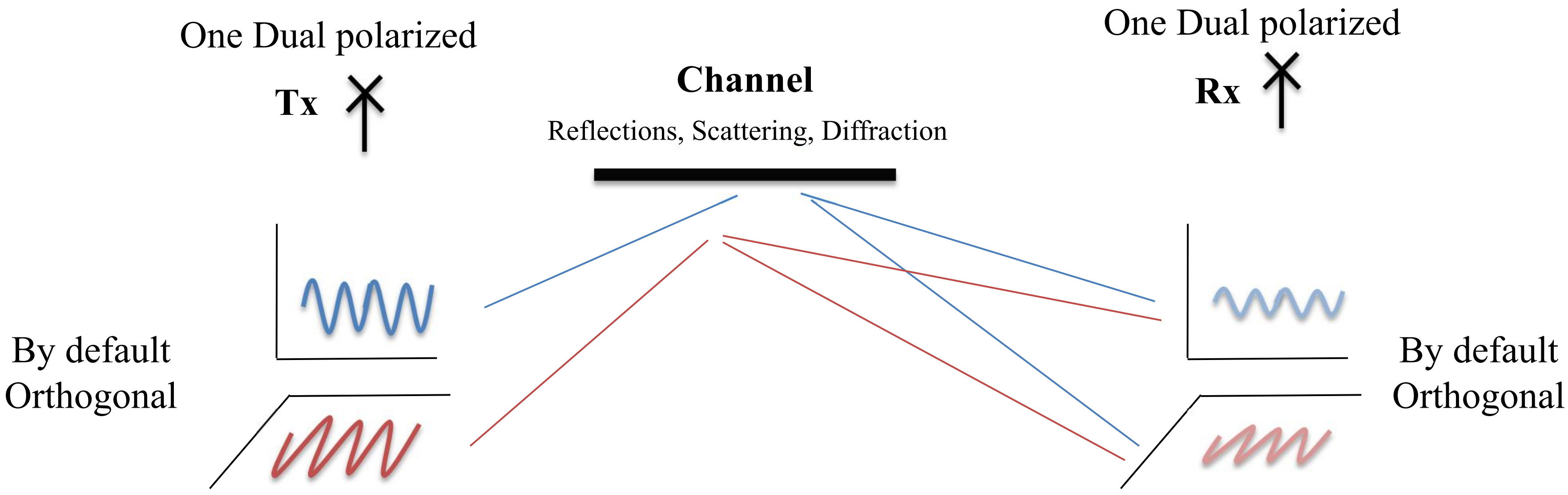
Motivation

- *Orthogonal spacetime block codes provide multiple gains, however, maximal rate designs in MIMOs are difficult to construct*
- *Dual-polarized antennas offer a good quality of service through reliable communication by mitigating multipath effects*
- *Efficient codes for multiple dual-polarized antennas*
- *Polarization diversity gain along with space & time diversities*
- *Low-complexity decoding (de-coupled decoding)*

Polarization Diversity Gain



One Dual polarized

Tx

Channel

Reflections, Scattering, Diffraction

One Dual polarized

Rx

By default
Orthogonal

By default
Orthogonal

- Space & Cost Effective
- Optimal Channel Separation

Low intensity indicates that the received signal is generally different from what is transmitted.

[Oestges C., Clerckx B., Guillaud M. and Debbah M., IEEE Tran. of Wireless Comm., 2008]

Quick Review

Quaternions A quaternion is a generalization of the concept of complex numbers defined over a basis of **non-commuting** elements $\{1, i, \mathrm{j}, \mathrm{k}\}$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$i^2 = j^2 = k^2 = -1$$

$$i\,j = k = -j\,i$$

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$$\boxed{\boldsymbol{Q} \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C}}$$

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Quaternion conjugate $$q^Q = z^*{}_1 - j\,z_2{}^* \longrightarrow qq^Q = |q|^2 = q^Q q$$

$$\downarrow$$

For normalized signals $$|q| = 1$$

Realization

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"Polarization state of electromagnetic waves can be represented by quaternions "

[Isaeva O. M. and Sarytchev V. A. , in Proc. 2nd IEEE Topical Symposium of Combined Optical-Microwave Earth and Atmosphere Sensing, Atlanta, US, April 1995, pp. 195–196.]

Quaternion Orthogonal Codes (QODs)

A quaternion orthogonal code is an n x m matrix of quaternion elements which satisfy

$$\boldsymbol{Q}^{\boldsymbol{Q}}\boldsymbol{Q} = \sum_{k=1}^{n} |z_k|^2 \boldsymbol{I_n} = \lambda \boldsymbol{I_n}$$

[Seberry J., et. al., “The theory of quaternion orthogonal designs,” IEEE Trans. Signal Process., 2008.]

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Symmetric-Pair Design:

Two complex orthogonal codes ***A*** and ***B*** form a symmetric-pair design if $\boldsymbol{A^H B}$ or $\boldsymbol{B^H A}$ is symmetric.

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QOD → $\boxed{\boldsymbol{Q} = \boldsymbol{A} + \boldsymbol{B}j}$

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System Model (Dual Polarized Antennas)

Brief Example

$$\boldsymbol{C} = \begin{array}{cc} \text{Antenna 1} & \text{Antenna 2} & \\ \left[\begin{matrix} z_1 \\ -z_2^{*} \end{matrix}\right. & \left.\begin{matrix} z_2 \\ z_1^{*} \end{matrix}\right] & \begin{matrix} \text{Time Slot 1} \\ \text{Time Slot 2} \end{matrix} \end{array}$$

$$\boxed{\boldsymbol{C}^H\boldsymbol{C} = \lambda\boldsymbol{I} = \boldsymbol{C}\boldsymbol{C}^H}$$

Almouti Scheme

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QODs

Two dual-polarized Antennas

$$Q = \begin{bmatrix} z_1 + z_2 j & z_2 + z_1 j \\ -z_2^* + z_1^* j & z_1^* - z_2^* j \end{bmatrix}$$

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$$\boxed{\boldsymbol{Q}^Q\boldsymbol{Q} = \lambda\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{Q}^Q}$$

$$\boldsymbol{C_q} = \begin{bmatrix} z_1 & z_2 & z_2 & z_1 \\ -z_2^* & z_1^* & z_1^* & -z_2^* \end{bmatrix}$$

$$\boxed{\boldsymbol{C_q}^H\boldsymbol{C_q} \neq \lambda\boldsymbol{I}}$$

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quasi-Orthogonal Code:

Four single polarized antennas are transmitting two complex symbols in two time slots.

Quaternion Orthogonal Codes (QODs)

An efficient way to generate square CODs of order $2^l \times 2^l$ is

$$\begin{bmatrix} G_{2^{l-1}}(z_1,\ldots,z_l) & z_{l+1} I_{2^{l-1}} \\ -z_{l+1}^{*} I_{2^{l-1}} & G_{2^{l-1}}^{H}(z_1,\ldots,z_l) \end{bmatrix} \qquad (1)$$

(Liang X. B., *Tran. of Inf. Theo.*, 2003)

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Theorem 1:

Two CODs ***A*** *and* ***B*** *of the form* (1), *where* ***B*** *is obtained by permuting the columns of* ***A****, satisfy both symmetry and complex amicable properties. Consequently,* ***Q*** = ***A*** + ***B*** *j is a QOD of rate* $(l+1)/2^l$.

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Non-uniqueness

Lemma 1:

For a given square COD $G_{2^{l-1}}(z_1,\ldots,z_l)$*, the matrix*

$$\boldsymbol{Q} = \begin{bmatrix} G_{2^{l-1}}(z_1,\ldots,z_l) + z_{l+1} I_{2^{l-1}}\, j \\ -z_{l+1}^{\ *} I_{2^{l-1}} + G_{2^{l-1}}^{\ \ H}(z_1,\ldots,z_l)\, j \end{bmatrix}$$

is a QOD of rate $(l+1)/2^l$.

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$$\boldsymbol{G}_2 = \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix}$$

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$$\boldsymbol{A}_4 = \begin{bmatrix} z_1 & z_2 & & \\ -z_2^{*} & z_1^{*} & & \\ & & z_1^{*} & -z_2 \\ & & z_2^{*} & z_1 \end{bmatrix}$$

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↙

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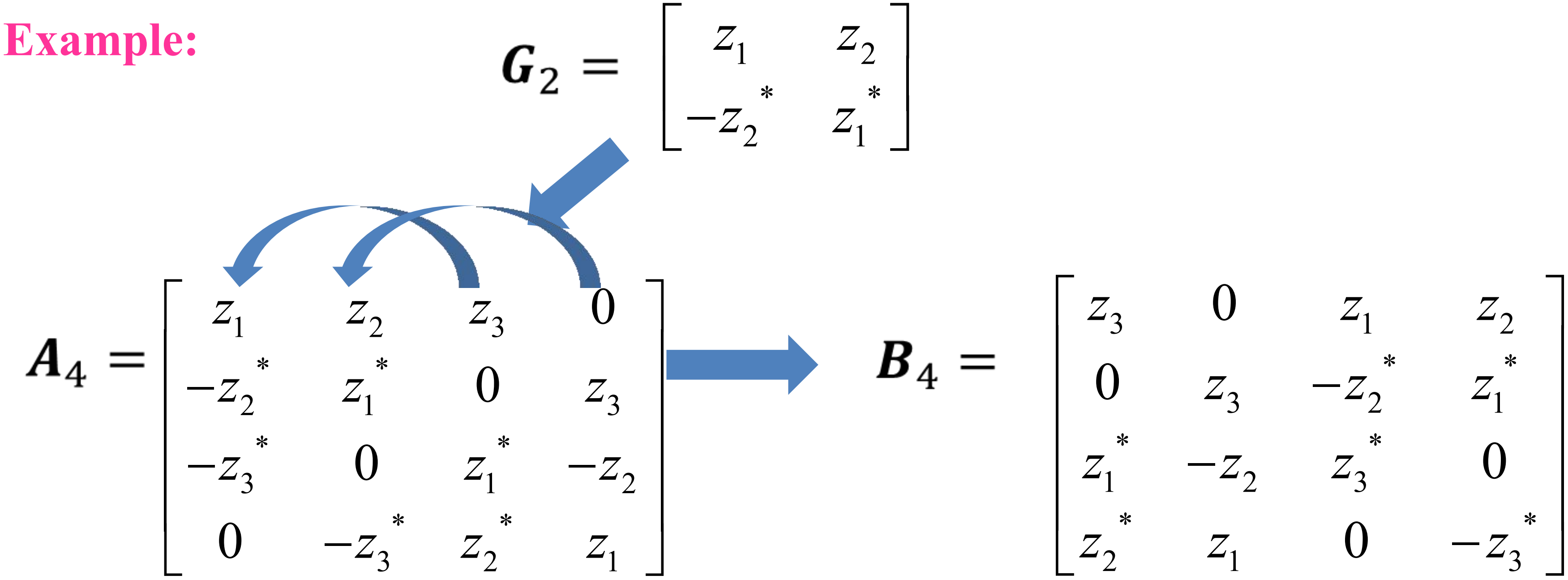

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Code rate = 3/4

$$\boxed{\boldsymbol{Q}_4{}^Q \boldsymbol{Q}_4 = \lambda\, \boldsymbol{I}_4}$$

System Model (Dual Polarized Antennas)

Consider a MISO tranmission dual-polarized system ($N_t \times 1$). The system model is

$$\boldsymbol{R} = C^{-1}(\boldsymbol{C}_q \boldsymbol{H} + \boldsymbol{N})$$

The operator C performs following operation

$C(z_1 + z_2 j) = [z_1, z_2]$

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Rayleigh Fading Channel

(zero-mean and unit variance)

$$\boldsymbol{H} = \begin{bmatrix} H^{(1)} \\ \vdots \\ H^{(N_t)} \end{bmatrix} \quad \text{where} \quad H^{(m)} = \begin{bmatrix} h_{11}^{(m)} & h_{12}^{(m)} \\ h_{21}^{(m)} & h_{22}^{(m)} \end{bmatrix}$$

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White noise

(Gaussian RVs iid zero mean and identical variance)

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The complex matrix C_q is obtained by decomposing a QOD with odd columns representing symbols transmitted through one polarization while even columns contain symbols transmitted through orthogonal polarization.

Quaternion Orthogonal Codes (QODs)

Low Complexity Decoder:

$$\min_{z_u}\left(\left\|R - C^{-1}(C_q H)\right\|^2\right) = \Gamma^{(1)} + \Gamma^{(2)} + \Gamma^{(3)}$$

$$\Gamma^{(1)} = tr\left(R^Q R\right), \quad \Gamma^{(2)} = -2\,\mathrm{Re}\left(tr\left(R^Q C^{-1}(C_q H)\right)\right)$$

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Lemma 2:

The ML-decoding rule for both constructions given in Theorem 1 and Lemma 1, simplifies to

$$\min_{z_u}\left(\left\|R - C^{-1}(C_q H)\right\|^2\right) = \Gamma^{(2)} = -2\,\mathrm{Re}\left(tr\left(R^Q C^{-1}(C_q H)\right)\right)$$

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Decoupled Decoder:

Code rate = 3/4

$$\min_{z_u} (-2\,\mathrm{Re}\{r_1^Q z_1 g_{12} + r_2^Q z_1^* g_{34} + r_3^Q z_1^* g_{56} + r_4^Q z_1 g_{78}\}),$$

$$\min_{z_u} (-2\,\mathrm{Re}\{r_1^Q z_2 g_{34} - r_2^Q z_2^* g_{12} - r_3^Q z_1 g_{78} + r_4^Q z_2^* g_{56}\}),$$

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where $g_{mn}=g_m+g_n j$ such that

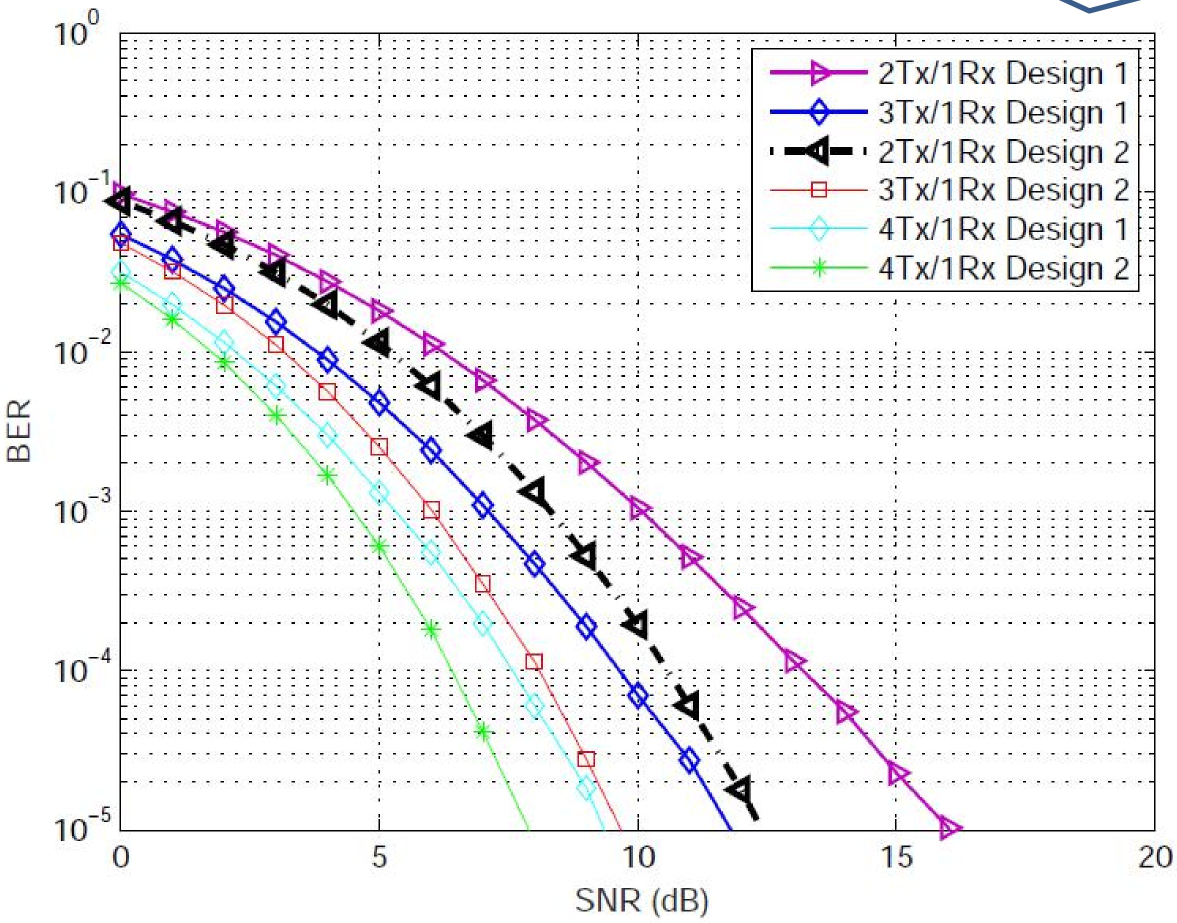
$$g_1=h_{11}^{(1)}+h_{21}^{(3)},\; g_2=h_{12}^{(1)}+h_{22}^{(3)}, g_3=h_{11}^{(2)}+h_{21}^{(4)}, g_4=h_{12}^{(2)}+h_{22}^{(4)},$$

$$g_5=h_{21}^{(1)}+h_{11}^{(3)}, g_6=h_{22}^{(1)}+h_{12}^{(3)}, g_7=h_{21}^{(2)}+h_{11}^{(4)}, g_6=h_{22}^{(2)}+h_{12}^{(4)}.$$

Quaternion Orthogonal Codes (QODs)

Performance Analysis:

Design 1 → Theorem 1
Design 2 → Lemma 1



Thought To Take Away !!!

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Have coeffee first and questions later ....

Conclusion

- Designs based on quaternions provide a feasible solution for dual-polarized antennas and easy to generate
- QODs exploit polarization diversity along with other diversities
- Decoupled decoding becomes an inherited characteristic of the approach
- Simulation results also confirm a performance up gradation in MIMOs against standard complex orthogonal or quasi-orthogonal codes which have other shortcomings