Symmetry Analysis And Exact Solutions of Semi-linear Heat Equation in Multi-Dimensions

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Motivation

1. Similarity solutions of nonlinear partial differential equations (PDEs)

2. Similarity reduction yield an ordinary differential equations (ODE) that may be intractable to solve explicitly

3. Alternative method based on Group-foliations able to yield explicit similarity solutions as well as non-similarity solutions

Semilinear Radial Heat Equation

$$u_t = u_{rr} + \frac{n-1}{r} u_r + k|u|^q u, \quad k = \pm 1$$

where r denotes the radial coordinate in $n > 1$ dimensions. This equation describes radial heat flow with a nonlinear heat source/sink term depending on power $q \neq 0$. The stability of solutions is determined by the cofficient k.

For $k = -1$, all smooth solutions $u(t, r)$ asymptotically approach a similarity form $u = t^{-1/q} U(r/\sqrt{t})$ exhibiting global dispersive behaviour $u \to 0$ as $t \to \infty$ for all $r \geq 0$.

For $k = +1$, some solutions $u(t, r)$ exhibit a blow-up behaviour $u \to \infty$ given by a similarity form $u = (T - t)^{-1/q} U(r/\sqrt{T - t})$ as $t \to T < \infty$.

Semilinear Radial Heat Equation

For finding exact solutions it is easier to work on a simpler equation

$$u_t = u_{rr} + \frac{n-1}{r} u_r + k u^{q+1}, \quad k = \pm 1$$

Its algebra of point symmetries is two-dimensional.

1. Time Translation

$$\frac{\partial}{\partial t}$$

2. Scaling Symmetry

$$t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2}{q} u \frac{\partial}{\partial u}$$

Classical Similarity Reduction

The similarity reduction relative to scaling symmetry reduces heat equation into the nonlinear ODE

$$U'' + ((n-1)\xi^{-1} + \tfrac{1}{2}\xi)U' - \tfrac{1}{2}qU + U|U|^q = 0$$

where

$$\xi = \begin{cases} r/\sqrt{t}, & k = -1 \\ ir/\sqrt{T-t}, & k = 1 \end{cases}$$

Remarks:

1. This ODE has no point symmetry.
2. This ODE has no quadratic first integrals
$\psi = A(\xi, U) + B(\xi, U)U' + C(\xi, U)U'^2$.
3. Few exact solutions $U(\xi)$ are known to-date other than explicit constant solution $U = (q/2)^{1/2}$.

Lie's idea of Group-Foliation

1. Rewrite heat equation as an equivalent first-order PDE system whose independent variables are similarity invariants and whose dependent variables are similarity differential invariants.

2. Separation ansatz to find the exact solutions of the equivalent system.

3. Integrate a pair of ODEs to get corresponding solution of the heat equations.

Group-Resolving Equations

Invariants and differential invariants of scaling symmetry are

$$x = t/r^2, \quad v = u/r^p$$
$$G(x,v) = r^{2-p}u_t, \quad H(x,v) = r^{1-p}u_r$$

The integrability relation $D_t u_r = D_r u_t$ yield first-order PDE

$$(p-2)G - 2xG_x - H_x - pvG_v + HG_v - GH_v = 0$$

and heat equation yields another PDE

$$G - (p+n-2)H + 2xH_x + pvH_v - HH_v - kv^{q+1} = 0$$

The above system of first-order PDEs is called scaling group-resolving system.

Group-Resolving Equations

Lemma 1.
All similarity solutions of heat equation correspond to solutions of the scaling-group resolving system with the specific form

$$H = pv - 2xG.$$

Lemma 2.
There is a mapping between the solutions $u(t, r)$ of heat equation and the solutions $G(x, v), H(x, v)$ of its scaling-group resolving system. The map $u \to (G, H)$ is many-to-one and onto; its inverse $(G, H) \to u$ is one-to-many and onto. In particular, the inverse mapping consists of integrating a consistent pair of parametric first-order ODEs

$$u_t = r^{p-2} G(t/r^2, u/r^p) \quad u_r = r^{p-1} H(t/r^2, u/r^p)$$

Group-Resolving Equations

Group-Resolving System

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Features

1. Inhomogeneous term contains power nonlinearity v^{q+1}

2. Linear terms have scaling-homogenous form in v-derivatives

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Group-Resolving Equations

The general features of group-resolving system suggest us following two-term ansatz $a \neq 1$

$$G = g_1(x)v^a + g_2(x)v$$
$$H = h_1(x)v^a + h_2(x)v$$

The GH-system contains powers v^a, v^{2a-1} and v that needs to be balanced with inhomogeneous term v^{q+1}.

Case 1. $(a = q + 1)$
This case gives us a one-term solution i.e.

$$G = kv^{q+1}, \quad H = 0, \quad q \neq -1$$

Group-Resolving Equations

Case 2. $(2a = q + 2)$

The coefficients of different powers of v in GH-system gives an overdetermined system of equations

$$\begin{aligned}
(2+q)h_1^2 + 2k = 0&,\\
2h_1' + 4xg_1' + (2 + qh_2)g_1 + qh_1g_2 = 0&,\\
h_2' + 2xg_2' + 2g_2 = 0&\\
2xh_2' - h_2^2 - (n-2)h_2 + g_2 = 0&,\\
4xh_1' - (2(n-1) + (q+4)h_2)h_1 + 2g_1 = 0&
\end{aligned}$$

Group-Resolving Equations

Case 2. $(2a = q + 2)$

Using computer algebra system (CRACK), we obtain these three solutions

$$G = (4-n)\sqrt{\frac{-k(n-2)}{n-3}}\, v^{(n-3)/(n-2)}, \quad H = \frac{1}{4-n}G + (2-n)v,$$

$$q = \frac{2}{2-n} \neq -1, \;\; n \neq 2, 3, 4$$

$$G = \frac{3v}{3x+1} - \frac{3\sqrt{k}}{2v}, \quad H = \frac{2}{3}G - \frac{v}{2}, \quad q = -4, \;\; n = 5/2$$

$$G = \frac{3v\left(1 + \sqrt{-2kv}\right)}{3x+1}, \quad H = \frac{1}{6}(3x+1)G - \frac{3x-1}{3x+1}v, \;\; q = 2, \;\; n = 5/2$$

Group-Resolving Equations

Following the success of two-term ansatz we assumed a three-term ansatz, $a \neq b \neq 1$

$$G = g_1(x)v^a + g_2(x)v^b + g_3(x)v,$$
$$H = h_1(x)v^a + h_2(x)v^b + h_3(x)v$$

We obtain the following solutions

$$G = \frac{3}{4}\sqrt{-2k}\left(v - \frac{1}{\sqrt{-2k}}\right)^2, \; H = \frac{2}{3}G + v - \frac{2}{\sqrt{-2k}}, \; q = 2, \; n = 5/2$$
$$G = \frac{15}{4}\sqrt{-2k}\left(v + \frac{1}{\sqrt{-2k}}\right)^2, \; H = \frac{2}{15}G + v + \frac{2}{\sqrt{-2k}}, \; q = 2, \; n = 5/2$$

Invariant Solutions

Altogether we have six solutions of group-resolving system. This yields six explicit solutions of the heat equation. There are four solutions which are similarity solutions modulo time-translation.

1. $u = (-kq(t+c))^{-1/q}, \quad q \neq 0$

2. $$u = \left(\pm\sqrt{\frac{-k}{(n-2)(n-3)}} \left(\frac{r}{2} - \frac{(n-4)(t+c)}{r} \right) \right)^{n-2},$$
   $$q = \frac{2}{2-n} \neq -1, n \neq 2, 3, 4$$

3. $$u = \pm\frac{3(t+c-r^2)}{r(3(t+c)+r^2)\sqrt{-2k}}, \quad q = 2, \quad n = 5/2$$

4. $$u = \pm\frac{5(3(t+c)+r^2)}{r(15(t+c)+r^2)\sqrt{-2k}}, \quad q = 2, \quad n = 5/2$$

where c is an arbitrary constant.

Non-invariant Solutions

Two other solutions are non-similarity solutions

1. $u = \left(\pm\sqrt{k}\,(1 + c(3t + r^2)) \left(\frac{3t}{r} + r \right) \right)^{1/2}, \quad q = -4, \; n = 5/2$
2. $u = \pm \frac{5(3t + r^2)}{\left(r(15t + r^2) + c\sqrt{r}\right)\sqrt{-2k}}, \quad q = 2, \; n = 5/2$

where c is an arbitrary constant.

Another Interpretation of Radial Heat Equation

How to interpret solutions where n is non-integer?

The heat equation can be written in a different form

$$u_t = u_{rr} + (1-\nu)r^{-1}u_r + ku^{q+1}, \quad \nu = const, \quad k = const$$

in terms of a parameter $\nu = 2 - n$ which applies to non-integer values of n. This equation describes radial heat flow in $\mathbb{R}^2$ with an extra source/sink term given by $\nu u_r / r$. The two-dimensional heat integral is given as

$$H = \int_0^\infty urdr$$

Another Interpretation of Radial Heat Equation

The heat integral satisfy the radial flux equation

$$\frac{dH}{dt} = S + F + \nu \lim_{r \to 0} u$$

where

$$F = \lim_{r \to 0} (-ru_r)$$

defines the outward radial heat flux at the origin, and

$$S = k \int_0^\infty u^{q+1} r dr$$

gives the net amount of heating or cooling caused by the nonlinear source/sink term in the heat equation.

Conclusion

1. The limitation of similarity reduction is that we still need to solve reduced ODE if it is applied to the original nonlinear PDE whereas in group-foliations similarity solutions can be obtained in explicit form.

2. This method yields non-similarity solutions which are not invariant under any one-dimensional subgroup of the full symmetry group.

3. It can be applied to nonlinear PDEs with any group of point symmetries (any one-dimensional point symmetry group is equivalent to the scaling group).

4. In future, we plan to compare classical symmetry reduction to group-foliation method for a variety of PDEs such as nonlinear wave equation, nonlinear diffusion equations and evolution equations.

References

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